Coincidence and Similarity Isometries of Modules in Euclidean Space

Svenja Glied

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Svenja Glied

Fakultät für Mathematik
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1. Berichterstatter: Prof. Dr. Michael Baake
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Contents

Introduction

The concept of coincidence isometries of lattices was used to study grain boundaries of crystals already several decades ago [**12, 33, 40**]. With the discovery of quasicrystals in 1984, an extension of the "classic" results became necessary and made new algebraic methods desirable that provided a uniform approach to both the crystallographic and quasicrystallographic situation. A basis for this was established in [**2**]. Whereas the classification of grain boundaries in crystals and quasicrystals is closely related to the existence of coincidence sublattices of the underlying lattice of periods or the corresponding translation module [**2**], the concept of similarity isometry arises for example in the context of colour symmetries [**5**]. It is thus of interest to understand the corresponding groups of isometries.

For a free $\mathbb{Z}$-module $M \subset \mathbb{R}^d$ of finite rank that spans $\mathbb{R}^d$, an element $R \in \mathrm{O}(d, \mathbb{R})$ is called a *coincidence isometry of* M if RM and M are commensurate, written $RM \sim M$, which means that their intersection has finite subgroup index both in M and in RM. We let $\mathrm{OC}(M)$ denote the set of all coincidence isometries of M. More generally, a *similarity isometry of* M is an element $T \in \mathrm{O}(d, \mathbb{R})$ with $\alpha TM \sim M$ for some positive real number α. This definition was introduced for lattices in [**7**]. The set $\mathrm{OS}(M)$ of all similarity isometries of M obviously contains $\mathrm{OC}(M)$ as a subset.

For subrings S of the rings of integers of real algebraic number fields, we consider the similarity and coincidence isometries of free S-modules $\Gamma \subset \mathbb{R}^d$ of rank d that span $\mathbb{R}^d$. This certainly includes the crystallographic case $S = \mathbb{Z}$, where Γ is a lattice; cf. [**22**]. We show that the similarity isometries of Γ form a group that contains the coincidence isometries as a normal subgroup. The corresponding factor group (of similarity modulo coincidence isometries) is the direct sum of cyclic groups of prime power orders that divide d (Thm. 1.23). In particular, if the dimension d is a prime number p, the factor group is an elementary Abelian p-group. In the case of *S-modules over K in* $\mathbb{R}^d$ (cf. Def. 1.26), where K is the quotient field of S, the factor group is either trivial or an elementary Abelian 2-group, depending on the parity of d. This includes settings relevant in quasicrystallography, for instance the standard icosahedral modules and the rings of cyclotomic integers in complex cyclotomic fields. Another example that fits this setting is the cubic lattice $\mathbb{Z}^3$. The group of its coincidence isometries coincides with the group of similarity isometries. In the second chapter, the structure of the group $\mathrm{OC}(\mathbb{Z}^3) = \mathrm{O}(3, \mathbb{Q})$ is investigated. Due to the fact that $\mathrm{O}(3, \mathbb{R}) = \mathrm{SO}(3, \mathbb{R}) \times C_2$ is a direct product, we restrict our attention to the coincidence rotations $\mathrm{SOC}(\mathbb{Z}^3) = \mathrm{SO}(3, \mathbb{Q})$. The situation here is more complex than for $\mathrm{SO}(2, \mathbb{Q})$

since $\mathrm{SO}(3,\mathbb{Q})$ is non-Abelian. Generally, the 2-dimensional case is, in contrast to the 3-dimensional one, rather well-understood [**2, 32, 11**].

Studying the structure of $\mathrm{SO}(3,\mathbb{Q})$ leads to the task of determining its subgroups. The classification of finite subgroups of $\mathrm{SO}(3,\mathbb{Q})$ is well known (cf. Section 2.3). They are comprised of the trivial group, the cyclic groups C_2, C_3, C_4, C_6, the dihedral groups D_2, D_3, D_4, D_6 as well as the alternating group A_4 and the symmetric group S_4. These groups all arise as the rotation symmetry group of a regular pyramid, a regular prism or a Platonic solid.

As a step towards determining the finitely generated subgroups of $\mathrm{SO}(3,\mathbb{Q})$, certain 2-generator subgroups are considered. More precisely, we classify those subgroups that are generated by two rotations of finite order about rotation axes that are themselves separated by an angle of finite order, i.e. an angle that is a rational multiple of π. These groups are called *generalised dihedral groups* in [**37**]. They appear as orientation groups in the theory of tilings of Euclidean 3-space. For example, the congruent triangular prisms in the "quaquaversal tiling" constructed in [**15**] appear in an infinite number of orientations. There exists a generalised dihedral group such that the orientations of any two prisms in that tiling are related by an element of this group.

The generalised dihedral subgroups of $\mathrm{SO}(3,\mathbb{R})$ are mostly free products or amalgamated free products of cyclic or dihedral groups and thus generally infinite, the only exceptions being the rotation symmetry groups of certain polyhedra. We use Cayley's parametrisation of $\mathrm{SO}(3,\mathbb{R})$ to represent rotations by quaternions and combine group theoretic results on (amalgamated) free products of groups with previous results on generalised dihedral subgroups of $\mathrm{SO}(3,\mathbb{R})$ (cf. Section 2.4) to obtain our main result of the second chapter (Thm. 2.42). It classifies the generalised dihedral subgroups of $\mathrm{SO}(3,\mathbb{Q})$ as the finite subgroups of $\mathrm{SO}(3,\mathbb{Q})$ except the alternating group A_4.

Finally, since the group of coincidence rotations of the standard icosahedral modules is $\mathrm{SO}(3,\mathbb{Q}(\tau))$, where $\tau = (1+\sqrt{5})/2$ is the golden ratio, this group is of interest as well. We classify the finite generalised dihedral subgroups of $\mathrm{SO}(3,\mathbb{Q}(\tau))$ in Theorem 2.48. In contrast to the rational case, $\mathrm{SO}(3,\mathbb{Q}(\tau))$ contains generalised dihedral groups that are infinite. Some examples of these infinite groups are presented. We conclude by stating the analogous classification for $\mathbb{Q}(\sqrt{2})$ and $\mathbb{Q}(\sqrt{3})$ instead of $\mathbb{Q}(\tau)$.

CHAPTER 1

Coincidence and Similarity Isometries for Modules

1.1. $\mathbb{Z}$-modules

We begin by recalling some well-known facts on Abelian groups. If M is an Abelian group and $N \subset M$ a subgroup of finite index $[M : N] = k$, then a direct consequence of Lagrange's Theorem is that kM is a subgroup of N.

LEMMA 1.1. *Let M be a free $\mathbb{Z}$-module of finite rank.*

(1) *If N is a submodule of M, then N is also a free $\mathbb{Z}$-module with* $\operatorname{rank}(N) \leqslant \operatorname{rank}(M)$.
(2) *If N is a submodule of M of finite index, then N has the same rank as M.*

PROOF. Cf. [**1**, Thm. 6.2] for (1). If $[M : N] = k \in \mathbb{N}$, then $kM \subset N$. If $\{m_1, \dots, m_r\}$ is a $\mathbb{Z}$-basis of M, then $\{km_1, \dots, km_r\}$ forms a $\mathbb{Z}$-basis for the free module kM. Using (1) yields $\operatorname{rank}(M) = \operatorname{rank}(kM) \leqslant \operatorname{rank}(N) \leqslant \operatorname{rank}(M)$. $\square$

LEMMA 1.2. [**13**, Ch. 2, Lemma 6.1.1] *If M is a torsion-free Abelian group of rank r, and N is a subgroup which is also of rank r, then the subgroup index $[M : N]$ is finite and equals the absolute value of the determinant of the transition matrix from any $\mathbb{Z}$-basis of M to any $\mathbb{Z}$-basis of N.* $\square$

Together with Lemma 1.1(2), this gives the following equivalence.

LEMMA 1.3. *Let $M \subset \mathbb{R}^d$ be a free $\mathbb{Z}$-module of finite rank and let $N \subset M$ be a submodule. Then the index $[M : N]$ is finite if and only if* $\operatorname{rank}(N) = \operatorname{rank}(M)$. $\square$

Two free $\mathbb{Z}$-modules $M, M' \subset \mathbb{R}^d$ of finite rank are called *commensurate* if their intersection has finite (subgroup) index both in M and in M'. In this case we write $M \sim M'$. If $M \sim M'$, then $M \cap M'$ is a free $\mathbb{Z}$-module with $\operatorname{rank}(M) = \operatorname{rank}(M') = \operatorname{rank}(M \cap M')$ by Lemma 1.1. Most results of this chapter are published in [**22, 21**].

LEMMA 1.4. *Commensurateness of free $\mathbb{Z}$-modules of finite rank r contained in $\mathbb{R}^d$ is an equivalence relation.*

PROOF. Reflexivity and symmetry are clear by definition. For the transitivity, let $M_1 \sim M_2$ and $M_2 \sim M_3$. In particular, the indices

$$s_{12} = [M_2 : (M_1 \cap M_2)] \quad \text{and} \quad s_{23} = [M_2 : (M_2 \cap M_3)]$$

are finite. One obtains $s_{12}M_2 \subset (M_1 \cap M_2)$ and $s_{23}M_2 \subset (M_2 \cap M_3)$. Since M_1, M_3 and $s_{12}s_{23}M_2$ are free $\mathbb{Z}$-modules of rank r in $\mathbb{R}^d$, Lemma 1.3 together with the relation

$$s_{12}s_{23}M_2 \subset (M_1 \cap M_2 \cap M_3)$$

now implies that $M_1 \cap M_2 \cap M_3$ is of finite index both in M_1 and M_3. As a consequence, one obtains $M_1 \sim M_3$. □

DEFINITION 1.5. Let $M \subset \mathbb{R}^d$ be a free $\mathbb{Z}$-module of finite rank. Then $R \in \mathrm{O}(d, \mathbb{R})$ is called a *coincidence isometry of* M, if $RM \sim M$. Define

$$\mathrm{OC}(M) := \{\, R \in \mathrm{O}(d, \mathbb{R}) \mid RM \sim M \,\}.$$

Furthermore, an element $R \in \mathrm{O}(d, \mathbb{R})$ is called a *similarity isometry of* M, if there exists a positive real number α such that $\alpha RM \sim M$. Define

$$\mathrm{OS}(M) := \{\, R \in \mathrm{O}(d, \mathbb{R}) \mid \alpha RM \sim M \text{ for some } \alpha \in \mathbb{R}_+ \,\}.$$

The set of similarity isometries of M obviously contains the set of coincidence isometries. Alternatively to the definition of $\mathrm{OS}(M)$ above, one has

$$\mathrm{OS}(M) = \{\, R \in \mathrm{O}(d, \mathbb{R}) \mid \beta RM \subset M \text{ for some } \beta \in \mathbb{R}_+ \,\}, \tag{1.1}$$

where $\mathbb{R}_+$ denotes the set of positive real numbers. Namely, if $R \in \mathrm{O}(d, \mathbb{R})$ with $\alpha RM \sim M$ for a suitable positive real number α, the index $[\alpha RM : (M \cap \alpha RM)] = k$ is finite, and hence $k\alpha RM \subset (M \cap \alpha RM) \subset M$ with $k\alpha \in \mathbb{R}_+$. Conversely, let $\beta \in \mathbb{R}_+$ with $\beta RM \subset M$. Since M and βRM are of the same finite rank, the index $[M : \beta RM] = \ell$ is finite by Lemma 1.3. Hence $\beta RM \sim M$, which proves (1.1). Therefore, $\mathrm{OS}(M)$ consists of all linear isometries that arise from similarity mappings of M into itself. We call a submodule of M of the form βRM as above a *similar submodule of* M. (For similar submodules in four dimensions, see [**9**].)

To illustrate these notions, we consider the square lattice $\mathbb{Z}^2$. Its coincidence isometries are precisely the orthogonal matrices with rational entries (cf. [**2**] or Corollary 1.18), i.e.

$$\mathrm{OC}(\mathbb{Z}^2) = \mathrm{O}(2, \mathbb{Q}).$$

To see this, take an arbitrary $R \in \mathrm{O}(2, \mathbb{Q})$ and denote by m the greatest common divisor of all $k \in \mathbb{N}$ such that kR is an integral matrix. Clearly, mR is then integral. We have to show that $R\mathbb{Z}^2$ and $\mathbb{Z}^2$ are commensurate. They both have rank 2 and both contain $mR\mathbb{Z}^2$ as a submodule, which also has rank 2. Hence,

$$mR\mathbb{Z}^2 \subset (R\mathbb{Z}^2 \cap \mathbb{Z}^2)$$

implies that $R\mathbb{Z}^2 \cap \mathbb{Z}^2$ has rank 2 by Lemma 1.1(1). Due to Lemma 1.3, the indices $[\mathbb{Z}^2 : (R\mathbb{Z}^2 \cap \mathbb{Z}^2)]$ and $[R\mathbb{Z}^2 : (R\mathbb{Z}^2 \cap \mathbb{Z}^2)]$ are finite, which means $R\mathbb{Z}^2 \sim \mathbb{Z}^2$. If conversely one entry of $R = (r_{ij})$ is irrational, say $r_{k\ell}$, consider the $\mathbb{Z}$-span of the image of the ℓ-th canonical basis vector Re_ℓ. One easily verifies that it does not contain any nonzero element of $\mathbb{Z}^2$. Setting $H := R\mathbb{Z}^2 \cap \mathbb{Z}^2$, this implies that infinitely many elements of $R\mathbb{Z}^2$ do not belong to H. More precisely, let $g, g' \in \mathbb{Z}Re_\ell$ with $g \neq g'$. Then $g - g' \in \mathbb{Z}Re_\ell$ does not

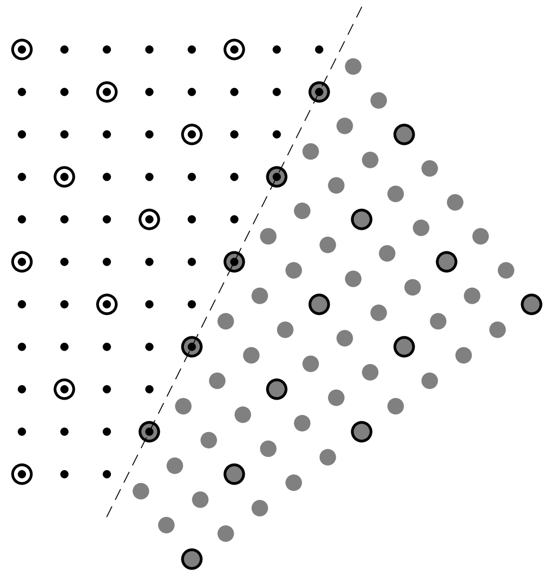

FIGURE 1.1. Planar "twin" with angle $\arctan(-4/3)$

belong to H, implying that the corresponding two cosets of H in $R\mathbb{Z}^2$ do not coincide. Thus, the factor group $R\mathbb{Z}^2/H$ contains infinitely many elements. This shows $R\mathbb{Z}^2 \not\sim \mathbb{Z}^2$ and hence R is not a coincidence isometry of $\mathbb{Z}^2$.

Figure 1.1 shows a grain boundary of a so-called "planar twin" as it could arise in a crystal. On the left hand side, there is a patch of the square lattice, whereas on the right hand side, there is a rotated copy of the square lattice. The rotation angle of the corresponding rotation R is $\arctan(-4/3)$. The dotted line indicates the actual grain boundary where there are common points of both lattices. The black circles indicate the common sublattice. It has subgroup index 5 in $\mathbb{Z}^2$ and in $R\mathbb{Z}^2$. Hence R is a coincidence rotation of the square lattice.

It is well known [**34**] that every orientation-reversing element of the orthogonal group $\mathrm{O}(2,\mathbb{R})$ can uniquely be written as the product of a rotation around the origin and the reflection in the x-axis, hence

$$\mathrm{O}(2,\mathbb{R}) = \mathrm{SO}(2,\mathbb{R}) \rtimes C_2,$$

where C_2 is the cyclic group of order 2 generated by the reflection $\left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$. We parametrise the Euclidean plane by the complex numbers $\mathbb{C}$. Then, every rotation by an angle θ about the origin corresponds to the multiplication with the element $e^{i\theta}$ of the unit circle $\mathbb{S}^1$. One has $\mathrm{SO}(2,\mathbb{R}) \simeq \mathbb{S}^1$. Now

$$\mathrm{O}(2,\mathbb{R}) = \mathbb{S}^1 \rtimes C_2,$$

where C_2 is generated by the complex conjugation in $\mathbb{C}$. To illustrate the notion of similarity isometry, we therefore restrict our attention to the similarity rotations of $\mathbb{Z}^2$. By

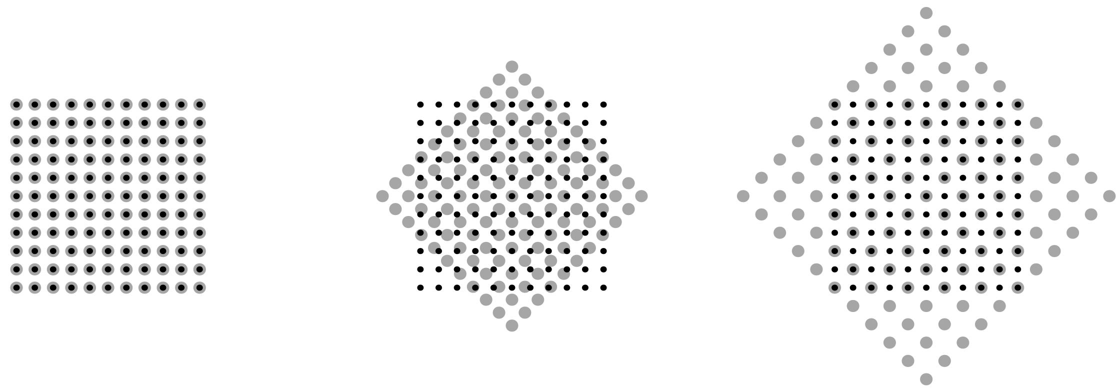

FIGURE 1.2. Similarity rotation of the square lattice $\mathbb{Z}^2$

the above identification, $\mathbb{Z}^2$ corresponds to the ring of Gaussian integers $\mathbb{Z}[i]$, where i is the imaginary unit. The group of similarity rotations of $\mathbb{Z}^2$ consists precisely of the set of $\mathbb{Z}^2$-directions,

$$\mathrm{SOS}(\mathbb{Z}^2) = \{\, a/|a| \mid 0 \neq a \in \mathbb{Z}[i] \,\}. \tag{1.2}$$

This is mainly due to the fact that $\mathbb{Z}[i]$ is a ring; compare Lemma 1.8 for a proof. For example, the multiplication by $(1+i)/|1+i|$ results in a rotation T by $\pi/4$. A subsequent multiplication by $\sqrt{2} = |1+i|$ gives a submodule of $\mathbb{Z}[i]$, because

$$\sqrt{2} \cdot ((1+i)/\sqrt{2})\, \mathbb{Z}[i] \subset \mathbb{Z}[i].$$

Therefore, one finds $\sqrt{2}T\mathbb{Z}^2 \sim \mathbb{Z}^2$. This similarity rotation is illustrated in Figure 1.2. Since $T \notin \mathrm{SO}(2,\mathbb{Q})$, it is an example of a similarity rotation that is not a coincidence rotation.

LEMMA 1.6. *Let $M_1, M_2 \subset \mathbb{R}^d$ be free $\mathbb{Z}$-modules of finite rank, and let $R \in \mathrm{GL}(d,\mathbb{R})$. Then $M_1 \sim M_2$ implies $RM_1 \sim RM_2$.*

PROOF. One easily verifies that $RM_i/(RM_1 \cap RM_2) \simeq M_i/(M_1 \cap M_2)$ for $i \in \{1,2\}$. Hence $[RM_i : (RM_1 \cap RM_2)] = [M_i : (M_1 \cap M_2)] < \infty$, which implies $RM_1 \sim RM_2$. $\square$

LEMMA 1.7. *Let $M \subset \mathbb{R}^d$ be a free $\mathbb{Z}$-module of finite rank. The sets $\mathrm{OS}(M)$ and $\mathrm{OC}(M)$ are subgroups of $\mathrm{O}(d,\mathbb{R})$.*

PROOF. Let $R, S \in \mathrm{OS}(M)$. Due to Equation (1.1), there exist positive real numbers α, β with $\alpha RM \subset M$ and $\beta SM \subset M$. Hence $RS \in \mathrm{OS}(M)$, because

$$\alpha\beta RSM = \alpha R(\beta SM) \subset \alpha RM \subset M.$$

One also has $M \subset \alpha^{-1}R^{-1}M$, implying that the group index $[\alpha^{-1}R^{-1}M : M] = s$ is finite by Lemma 1.3. Thus $s\alpha^{-1}R^{-1}M \subset M$ is also of finite index. This shows $R^{-1} \in \mathrm{OS}(M)$. For the group property of $\mathrm{OC}(M)$ let $R_1, R_2 \in \mathrm{OC}(M)$. By Lemma 1.6, $R_2M \sim M$

yields $M \sim R_2^{-1}M$, and hence $R_1M \sim R_1R_2^{-1}M$. Now $M \sim R_1M \sim R_1R_2^{-1}M$ implies $M \sim R_1R_2^{-1}M$, because commensurateness is transitive by Lemma 1.4. □

Let us briefly turn to the subgroup of orientation preserving similarity isometries $\mathrm{SOS}(M) \subset \mathrm{OS}(M)$, which are by definition those similarity isometries R with $\det(R) = 1$. For planar lattices, these similarity rotations are rather well understood; cf. [**11**]. A free $\mathbb{Z}$-module in an algebraic number field K of degree n is called *full*, if it contains n linearly independent elements over $\mathbb{Q}$. If a full $\mathbb{Z}$-module in K is a ring and contains the number 1, it is called an *order of* K. Any order of K is contained in the ring of algebraic integers $\mathcal{O}_K$ of K, which is itself an order. Hence $\mathcal{O}_K$ is also called the *maximal order of* K; cf. [**13**]. For any ring A we set $A^\bullet = A \setminus \{0\}$.

In the following results on orders of imaginary algebraic number fields, we once again parametrise the Euclidean plane by $\mathbb{C}$ and use $\mathrm{SO}(2,\mathbb{R}) \simeq \mathbb{S}^1$.

Lemma 1.8. *Let K be an imaginary algebraic number field and let $\mathcal{O}$ be an order of K. Then*

$$\mathrm{SOS}(\mathcal{O}) = \big\{a/|a| \big| a \in \mathcal{O}^\bullet\big\}.$$

Proof. For $0 \neq a \in \mathcal{O}$, one has $|a|(a/|a|)\mathcal{O} \subset \mathcal{O}$, because $\mathcal{O}$ is a ring. Therefore $a/|a| \in \mathrm{SOS}(\mathcal{O})$. Conversely, let $r \in \mathrm{SOS}(\mathcal{O})$, meaning $r \in \mathbb{S}^1$ with $\lambda r\mathcal{O} \subset \mathcal{O}$ for some $\lambda \in \mathbb{R}_+$. Since $1 \in \mathcal{O}$, this yields $\lambda r \in \mathcal{O}$, say $\lambda r = \beta$. Thus $|\lambda| = |\beta|$, because $r \in \mathbb{S}^1$. This shows that $r = \pm\beta/|\beta|$ is an $\mathcal{O}$-direction. □

There is a close connection between similar submodules of orders $\mathcal{O}$ of imaginary algebraic number fields K that arise from rotations and the principal ideals of these orders. The special cases where K is an n-th cyclotomic field of class number 1 and $\mathcal{O} = \mathbb{Z}[\zeta_n]$ (with ζ_n an n-th primitive root of unity), or where $\mathcal{O}$ is the multiplier ring of a non-generic[1] planar lattice, can be found in [**5**] and [**11**].

Theorem 1.9. *Let K be an imaginary algebraic number field and let $\mathcal{O}$ be an order of K. Then the similar submodules of $\mathcal{O}$ of the form $\alpha R\mathcal{O}$ with $R \in \mathrm{SO}(2,\mathbb{R})$ and $\alpha \in \mathbb{R}_+$ are precisely the nonzero principal ideals of $\mathcal{O}$. Moreover, one has*

$$[\mathcal{O} : \kappa\mathcal{O}] = |N(\kappa)|,$$

where N denotes the field norm of K.

Proof. Let $R \in \mathrm{SO}(2,\mathbb{R})$ and $\alpha \in \mathbb{R}_+$ with $\alpha R\mathcal{O} \subset \mathcal{O}$. Then $R \in \mathrm{SOS}(\mathcal{O})$ by Equation (1.1). Due to Lemma 1.8, there exists a nonzero $\delta \in \mathcal{O}$ such that $R = \delta/|\delta|$. Then $1 \in \mathcal{O}$ implies $\alpha\delta/|\delta| \in \mathcal{O}$. Hence $\alpha R\mathcal{O}$ is a principal ideal of $\mathcal{O}$. Conversely, for any nonzero $\kappa' \in \mathcal{O}$ one has $\kappa'\mathcal{O} \subset \mathcal{O}$, because $\mathcal{O}$ is a ring. Setting $R' = \kappa'/|\kappa'|$, one has $R' \in \mathrm{SOS}(\mathcal{O})$ by Lemma 1.8, and $|\kappa'|R'\mathcal{O} \subset \mathcal{O}$.

[1]Given a planar lattice Γ, the set $\mathrm{MR}(\Gamma) = \{\, a \in \mathbb{C} \mid a\Gamma \subset \Gamma \,\}$ forms a subring of $\mathbb{C}$ called the multiplier ring of Γ. The lattice Γ is called generic, if $\mathrm{MR}(\Gamma) = \mathbb{Z}$, and non-generic otherwise. In the latter case, the multiplier ring is an order in an imaginary quadratic field [**11**].

The second claim follows by a standard argument in Minkowski theory; cf. [**13**, Ch. 2, Sec. 3]. We merely sketch the proof. Let $\kappa \in \mathcal{O}^\bullet$ and let K have degree n over $\mathbb{Q}$. Furthermore, let $\{\gamma_1, \dots, \gamma_n\}$ be a $\mathbb{Z}$-basis of $\mathcal{O}$. Now consider a Minkowski representation $x(\mathcal{O})$ of $\mathcal{O}$. Then $x(\mathcal{O})$ is a lattice in $\mathbb{R}^n$ (cf. [**13**, Ch. 2, Sec. 3, Thm. 1]) and $\{x(\gamma_1), \dots, x(\gamma_n)\}$ is a basis of $x(\mathcal{O})$. On the other hand, $\kappa\mathcal{O}$ is also a $\mathbb{Z}$-module of rank n in K and thus $\{x(\kappa\gamma_1), \dots, x(\kappa\gamma_n)\}$ forms a basis for the lattice $x(\kappa\mathcal{O})$ in $\mathbb{R}^n$. Now, $\kappa\mathcal{O} \subset \mathcal{O}$ implies $x(\kappa\mathcal{O}) \subset x(\mathcal{O})$. Moreover, the map $x\colon K \longrightarrow \mathbb{R}^n$ is an injective group homomorphism. Hence $[\mathcal{O} : \kappa\mathcal{O}] = [x(\mathcal{O}) : x(\kappa\mathcal{O})]$. It therefore suffices to show $[x(\mathcal{O}) : x(\kappa\mathcal{O})] = N(\kappa)$. Since $x(\kappa\mathcal{O}) \subset x(\mathcal{O})$ is a sublattice, one has

$$[x(\mathcal{O}) : x(\kappa\mathcal{O})] = \frac{|\det(B_1)|}{|\det(B_2)|},$$

where B_1 is a basis matrix of $x(\kappa\mathcal{O})$ and B_2 is a basis matrix of $x(\mathcal{O})$ [**14**]. Denote by s the number of real isomorphisms of K into $\mathbb{C}$ and by $2t$ the number of complex isomorphisms of K into $\mathbb{C}$ (which come in complex conjugate pairs). Thus, $n = s + 2t$.

The multiplication with any element of $\mathbb{R}^n$ is a linear transformation of $\mathbb{R}^n$. By standard Minkowski theory, the matrix A of the multiplication with $x(\kappa)$ in terms of the $\mathbb{R}$-basis $\{e_1, \dots, e_s, e_{s+1}, ie_{s+1}, \dots, e_{s+t}, ie_{s+t}\}$ of $\mathbb{R}^n$ satisfies $\det(A) = N(\kappa)$. Since $x(\kappa\mathcal{O}) = x(\kappa)x(\mathcal{O})$, one has $B_1 = AB_2$. Consequently,

$$[x(\mathcal{O}) : x(\kappa\mathcal{O})] = \frac{|\det(A) \cdot \det(B_2)|}{|\det(B_2)|} = |N(\kappa)|.$$

□

1.2. S-modules

Let $S \subset \mathbb{R}$ be a ring with unity that is also a finitely generated $\mathbb{Z}$-module, hence a free $\mathbb{Z}$-module of finite rank r. Furthermore, let K be the field of fractions of S.

Throughout this section, let $\varGamma \subset \mathbb{R}^d$ be a free S-module of rank d that spans $\mathbb{R}^d$, meaning that it is the S-span of an $\mathbb{R}$-basis of $\mathbb{R}^d$. Besides the case $S = \mathbb{Z}$, where $\varGamma$ is a lattice in $\mathbb{R}^d$, this also covers many important examples relevant in quasicrystallography; cf. Example 1.27.

Remark 1.10. (1) Every element of S is an algebraic integer and K is a real algebraic number field. The ring $\mathcal{O}_K$ of algebraic integers of K is the (unique) maximal order of K, and hence is a $\mathbb{Z}$-module of rank $[K : \mathbb{Q}]$; cf. [**13**]. The algebraic number field K is of class number 1 if and only if the ring $\mathcal{O}_K$ is a principal ideal domain. It is easily verified that K is the quotient field of $\mathcal{O}_K$.

(2) S is integrally closed if and only if S is the ring of integers $\mathcal{O}_K$ in K.
(3) $\varGamma$ is a free $\mathbb{Z}$-module of rank rd.
(4) In fact, by standard results from algebra, the rings S as above are precisely the subrings of rings of integers in real algebraic number fields, which means the following. Given a ring S as above, then all its elements are algebraic integers and

its field of fractions K is a real algebraic number field. On the other hand, if S is a subring of the ring of integers $\mathcal{O}_{K'}$ of any real algebraic number field K', then S is a finitely generated $\mathbb{Z}$-module ; cf. [**23**] for more on this.

Rings S as above are precisely the subrings of rings of integers $\mathcal{O}_K$ in real algebraic number fields K. Namely, if S is as above with $\mathbb{Z}$-basis $\{\alpha_1, \dots, \alpha_r\}$, then every element of S is an algebraic integer by the explanations given in [**27**, Ch. 7]. With $S = \mathbb{Z}[\alpha_1, \dots, \alpha_r]$, the field of fractions K of S is $K = \mathbb{Q}(\alpha_1, \dots, \alpha_r)$. Consequently, K is a real algebraic number field; cf. [**27**, Ch. 5, Prop. 1.6]. On the other hand, if L is a real algebraic number field and S a subring of $\mathcal{O}_L$, one has the following. Since $\mathcal{O}_L$ is free of rank $n = [L : \mathbb{Q}]$ as a $\mathbb{Z}$-module, S is free of rank $r \leqslant n$ as a $\mathbb{Z}$-module; see [**27**, Ch. 3, Thm. 7.1]. In particular, S is finitely generated.

REMARK 1.11. Let $\varGamma_1, \varGamma_2$ be free S-modules of rank d that span $\mathbb{R}^d$. If $\varGamma_1$ and $\varGamma_2$ are commensurate, then $\varGamma_1 \cap \varGamma_2$ is a free $\mathbb{Z}$-module of rank rd (cf. p. 3) and it spans $\mathbb{R}^d$. Namely, if $\varGamma_1 \sim \varGamma_2$, then one has $m = [\varGamma_1 : (\varGamma_1 \cap \varGamma_2)] < \infty$. Hence $m\varGamma_1 \subset (\varGamma_1 \cap \varGamma_2)$, which implies that $\varGamma_1 \cap \varGamma_2$ contains an $\mathbb{R}$-basis of $\mathbb{R}^d$.

The following result is of fundamental importance and can be found in [**23**].

THEOREM 1.12. *Let $\varGamma_1, \varGamma_2 \subset \mathbb{R}^d$ be free S-modules of rank d that span $\mathbb{R}^d$. Further, let $B_1, B_2 \in \mathrm{GL}(d, \mathbb{R})$ be basis matrices of the S-modules $\varGamma_1$ and $\varGamma_2$, respectively. Then, one has*

$$\varGamma_1 \sim \varGamma_2 \iff B_2^{-1} B_1 \in \mathrm{GL}(d, K)\,.$$

PROOF. First, let $\varGamma_1 \sim \varGamma_2$. By Remark 1.11, the intersection $\varGamma_1 \cap \varGamma_2$ contains an $\mathbb{R}$-basis $\mathcal{B}$ of $\mathbb{R}^d$. Let $B \in \mathrm{GL}(d, \mathbb{R})$ be the associated matrix. Then there exist non-singular matrices $Z_1, Z_2 \in \mathrm{Mat}(d, S)$ such that

$$B_1 Z_1 = B = B_2 Z_2\,,$$

whence $B_2^{-1} B_1 = Z_2 Z_1^{-1} \in \mathrm{GL}(d, K)$ by the standard formula for the inverse of a matrix.

Conversely, if $B_2^{-1} B_1 \in \mathrm{GL}(d, K)$, then there is a nonzero number $s \in S$ such that $B = s B_2^{-1} B_1 \in \mathrm{Mat}(d, S)$. Setting $\varGamma' = \varGamma_1 \cap \varGamma_2$, the identity $sB_1 = B_2 B$ implies that $s\varGamma_1 \subset \varGamma' \subset \varGamma_1$. Since $s\varGamma_1$ and $\varGamma_1$ are both free $\mathbb{Z}$-modules of rank rd, one obtains $[\varGamma_1 : \varGamma'] < \infty$ by Lemma 1.3. By symmetry, one also has $[\varGamma_2 : \varGamma'] < \infty$. Hence, $\varGamma_1 \sim \varGamma_2$. □

DEFINITION 1.13. For an arbitrary element $R \in \mathrm{O}(d, \mathbb{R})$, define

$$\mathrm{scal}_{\varGamma}(R) = \{\, \alpha \in \mathbb{R} \mid \varGamma \sim \alpha R \varGamma \,\}.$$

Note that $\mathrm{OS}(\varGamma) = \{\, R \in \mathrm{O}(d, \mathbb{R}) \mid \mathrm{scal}_{\varGamma}(R) \neq \varnothing \,\}$.

REMARK 1.14. If $\beta \in \mathrm{scal}_{\varGamma}(R)$, then there exists a nonzero element $t \in \mathbb{Z}$ such that $t\beta R\varGamma \subset \varGamma$. For if $\beta \in \mathrm{scal}_{\varGamma}(R)$, then the index $[\beta R\varGamma : (\varGamma \cap \beta R\varGamma)] = t$ is finite and one has $t\beta R\varGamma \subset (\varGamma \cap \beta R\varGamma) \subset \varGamma$.

LEMMA 1.15. *Let $\varGamma \subset \mathbb{R}^d$ be a free S-module of rank d that spans $\mathbb{R}^d$. For all elements $\alpha \in \operatorname{scal}_\varGamma(R)$, one has $\alpha^d \in K$. Thus α is an algebraic number.*

PROOF. One has $\alpha R\varGamma \sim \varGamma$ by assumption. Let B be a basis matrix for $\varGamma$. Then αRB is a basis matrix for $\alpha R\varGamma$. By Theorem 1.12, one has $B^{-1}\alpha RB \in \mathrm{GL}(d,K)$, which immediately yields

$$\alpha^d = \pm\det(\alpha R) \in K.$$

Hence α is algebraic over K, which implies that $K(\alpha)$ is a finite field extension of K, and thus also of $\mathbb{Q}$. Therefore α is algebraic over $\mathbb{Q}$. □

Denoting by $\mathbb{R}^\bullet\,\mathrm{GL}(d,K)$ the group consisting of all elements of the form tH with $t \in \mathbb{R}^\bullet$ and $H \in \mathrm{GL}(d,K)$, there is the following consequence of Theorem 1.12.

COROLLARY 1.16. *Let $\varGamma \subset \mathbb{R}^d$ be a free S-module of rank d that spans $\mathbb{R}^d$. For any basis matrix $B_\varGamma$ of $\varGamma$, one has*

$$\mathrm{OS}(\varGamma) = \bigl(B_\varGamma\,(\mathbb{R}^\bullet\,\mathrm{GL}(d,K))\,B_\varGamma^{-1}\bigr) \cap \mathrm{O}(d,\mathbb{R}) \tag{1.3}$$

and

$$\mathrm{OC}(\varGamma) = \bigl(B_\varGamma\,\mathrm{GL}(d,K)B_\varGamma^{-1}\bigr) \cap \mathrm{O}(d,\mathbb{R}).$$

PROOF. Let $\{\gamma_1,\dots,\gamma_d\}$ be an S-basis of $\varGamma$ and denote by $B_\varGamma$ the associated matrix. For $R \in \mathrm{OS}(\varGamma)$, there exists a positive real number α with $\alpha R\varGamma \sim \varGamma$. The set $\{\alpha R\gamma_1,\dots,\alpha R\gamma_d\}$ is an S-basis of $\alpha R\varGamma$ with associated matrix $B_{\alpha R\varGamma} = \alpha RB_\varGamma$. Theorem 1.12 then implies that there exists an $H \in \mathrm{GL}(d,K)$ with $H = B_\varGamma^{-1}\alpha RB_\varGamma$. Thus, one has $R = B_\varGamma\alpha^{-1}HB_\varGamma^{-1}$.

If, on the other hand, $S \in \mathrm{O}(d,\mathbb{R})$ and $S = B_\varGamma\beta JB_\varGamma^{-1}$ for some $\beta \in \mathbb{R}^\bullet$ and some $J \in \mathrm{GL}(d,K)$, then one has

$$B_\varGamma^{-1}B_{\beta^{-1}S\varGamma} = B_\varGamma^{-1}\beta^{-1}SB_\varGamma \ \in \mathrm{GL}(d,K).$$

Theorem 1.12 therefore implies $\beta^{-1}S\varGamma \sim \varGamma$, which shows $S \in \mathrm{OS}(\varGamma)$. □

REMARK 1.17. Due to Corollary 1.16 above, every element $R \in \mathrm{OS}(\varGamma)$ can be written as $R = B_\varGamma\beta HB_\varGamma^{-1}$ with $\beta \in \mathbb{R}^\bullet$ and $H \in \mathrm{GL}(d,K)$. Theorem 1.12 yields $\beta R\varGamma \sim \varGamma$ and hence $\beta \in \operatorname{scal}_\varGamma(R)$, which implies that β is an algebraic number and that $\beta^d \in K$ by Lemma 1.15. But the set of all algebraic numbers is countable and so is $\mathrm{GL}(d,K)$. Therefore the group $\mathrm{OS}(\varGamma)$ is countable and, in particular, the subgroup $\mathrm{OC}(\varGamma)$ is countable as well. The explanations above imply that, in Corollaries 1.16 and 1.18, $\mathbb{R}^\bullet$ can be replaced by the set of all nonzero real numbers δ with $\delta^d \in K$.

COROLLARY 1.18. *Let $\varGamma \subset \mathbb{R}^d$ be a free S-module of rank d that spans $\mathbb{R}^d$ and let K be the field of fractions of S. If $\varGamma \subset K^d$, one has*

$$\mathrm{OS}(\varGamma) = \bigl(\mathbb{R}^\bullet\,\mathrm{GL}(d,K)\bigr) \cap \mathrm{O}(d,\mathbb{R})$$

and

$$\mathrm{OC}(\varGamma) = \mathrm{O}(d, K).$$

PROOF. The assumption $\varGamma \subset K^d$ yields $B_\varGamma \in \mathrm{GL}(d, K)$. Hence $B_\varGamma \,\mathrm{GL}(d, K) B_\varGamma^{-1}$ and $\mathrm{GL}(d, K)$ coincide, and the claim follows from Corollary 1.16. □

LEMMA 1.19. *Let $\varGamma \subset \mathbb{R}^d$ be a free S-module of rank d that spans $\mathbb{R}^d$. For $R \in \mathrm{OS}(\varGamma)$, the following assertions hold.*

(1) $b \cdot \mathrm{scal}_\varGamma(R) = \mathrm{scal}_\varGamma(R)$ *for all* $b \in K^\bullet$
(2) $r\varGamma \sim \varGamma$ *with* $r \in \mathbb{R}$ *implies* $r \in K$
(3) $\alpha\beta^{-1} \in K$ *for all* $\alpha, \beta \in \mathrm{scal}_\varGamma(R)$

PROOF. Let $\alpha \in \mathrm{scal}_\varGamma(R)$. Since K is the field of fractions of S, every nonzero element $b \in K$ can be written as $b = b_1/b_2$ with $b_1, b_2 \in S^\bullet$. By using Theorem 1.12, one easily finds

$$\frac{b_1}{b_2}\alpha R\varGamma \sim \frac{1}{b_2}\alpha R\varGamma \sim \frac{1}{b_2}\varGamma \sim \varGamma.$$

Hence $b\alpha R\varGamma \sim \varGamma$ by transitivity, yielding $b \cdot \mathrm{scal}_\varGamma(R) \subset \mathrm{scal}_\varGamma(R)$. Thus, one also has $b^{-1} \cdot \mathrm{scal}_\varGamma(R) \subset \mathrm{scal}_\varGamma(R)$, which proves (1). In order to show (2), let $u \in \mathbb{R}$ with $u\varGamma \sim \varGamma$. Due to Remark 1.14, there exists a nonzero integer k such that $ku\varGamma \subset \varGamma$. Let $\gamma \in \varGamma$ be represented in terms of an S-basis $\{\gamma_1, \ldots, \gamma_d\}$ of $\varGamma$ as $\gamma = \sum_{i=1}^d c_i\gamma_i$ with $c_i \in S$. On the other hand, $ku\gamma$ can be represented as $ku\gamma = \sum_{i=1}^d a_i\gamma_i$, where $a_i \in S$. Thus

$$\sum_{i=1}^d kuc_i\gamma_i = \sum_{i=1}^d a_i\gamma_i.$$

Since $\varGamma$ spans $\mathbb{R}^d$, $\{\gamma_1, \ldots, \gamma_d\}$ forms an $\mathbb{R}$-basis of $\mathbb{R}^d$. Therefore, one has $kuc_i = a_i$ for all $1 \leqslant i \leqslant d$, yielding $u = k^{-1}a_ic_i^{-1} \in K$. Finally, (3) is obtained from (2) as follows. By assumption, one has

$$\beta R\varGamma \sim \varGamma \sim \alpha R\varGamma.$$

Multiplying with $1/\beta$ gives $R\varGamma \sim \frac{\alpha}{\beta}R\varGamma$, which completes the proof. □

REMARK 1.20. As a direct consequence of Lemma 1.19, one has $\mathrm{scal}_\varGamma(R) = \alpha K^\bullet$ for any $\alpha \in \mathrm{scal}_\varGamma(R)$.

Define the map

$$\eta\colon\ \mathrm{OS}(\varGamma) \longrightarrow \mathbb{R}^\bullet/K^\bullet$$

by

$$R \longmapsto \mathrm{scal}_\varGamma(R).$$

This map is well-defined due to the fact that $\mathrm{scal}_\varGamma(R)$ is non-empty for $R \in \mathrm{OS}(\varGamma)$ and by Remark 1.20.

Proposition 1.21. *The map η is a group homomorphism with* $\mathrm{Ker}(\eta) = \mathrm{OC}(\Gamma)$.

Proof. Let $R, S \in \mathrm{OS}(\Gamma)$ and $\alpha \in \mathrm{scal}_\Gamma(R)$, $\beta \in \mathrm{scal}_\Gamma(S)$. We need to show that $\alpha\beta \in \mathrm{scal}_\Gamma(RS)$. Using Lemma 1.6, we get

$$\Gamma \sim \alpha R\Gamma \sim \alpha R(\beta S\Gamma) = \alpha\beta RS\Gamma.$$

Thus $\alpha\beta \in \mathrm{scal}_\Gamma(RS)$ and hence η is a group homomorphism. It only remains to show that $\mathrm{Ker}(\eta) = \mathrm{OC}(\Gamma)$. For $R \in \mathrm{OC}(\Gamma)$, the set $\mathrm{scal}_\Gamma(R)$ contains 1, thus $\eta(R) = \mathrm{scal}_\Gamma(R) = K^\bullet$ by Remark 1.20. Conversely, if $S \in \mathrm{Ker}(\eta)$, one has $\mathrm{scal}_\Gamma(S) = K^\bullet$, which indeed implies $S \in \mathrm{OC}(\Gamma)$. □

As the kernel of a group homomorphism, $\mathrm{OC}(\Gamma)$ is a normal subgroup of $\mathrm{OS}(\Gamma)$. The factor group $\mathrm{OS}(\Gamma)\,/\,\mathrm{OC}(\Gamma)$ is isomorphic to the image of η, which is a subgroup of $\mathbb{R}^\bullet/K^\bullet$ and thus Abelian. Furthermore, $\mathrm{OS}(\Gamma)\,/\,\mathrm{OC}(\Gamma)$ is countable by Remark 1.17. The corresponding result holds for the special case of orientation-preserving isometries by considering the restriction of η to $\mathrm{SOS}(\Gamma)$.

Our objective is to determine the structure of the factor group. Let us begin by a fully worked-out example. Consider the similarity and coincidence rotations of $\mathbb{Z}^2$. As seen before, one has $\mathrm{SOC}(\mathbb{Z}^2) = \mathrm{SO}(2, \mathbb{Q})$. On the other hand, one can identify $\mathbb{Z}^2$ with the Gaussian integers $\mathbb{Z}[i]$. Then, a rotation $R(\varphi)$ with rotation angle φ corresponds to a multiplication with the complex number $e^{i\varphi} \in (\mathbb{Q}(i) \cap \mathbb{S}^1) \simeq \mathrm{SOC}(\mathbb{Z}^2)$; see **[32]**. Using the fact that $\mathbb{Z}[i]$ is a unique factorisation domain, each coincidence rotation uniquely factorises as

$$e^{i\varphi} = \varepsilon \prod_{p\equiv 1(4)} \left(\frac{\omega_p}{\overline{\omega}_p}\right)^{n_p}, \tag{1.4}$$

where $\varepsilon = i^s$, $0 \leqslant s < 4$, is a unit in $\mathbb{Z}[i]$, $n_p \in \mathbb{Z}$ with only finitely many of them nonzero, p runs through the rational primes congruent to 1 (mod 4) and p factorises as $p = \omega_p\overline{\omega}_p$ in $\mathbb{Z}[i]$ with $\omega_p/\overline{\omega}_p$ not a unit. This shows that $\mathrm{SOC}(\mathbb{Z}^2)$ is a countably generated Abelian group. More precisely,

$$\mathrm{SOC}(\mathbb{Z}^2) = C_4 \times \mathbb{Z}^{(\aleph_0)},$$

where C_4 denotes the cyclic group of order 4 (here generated by i) and $\mathbb{Z}^{(\aleph_0)}$ stands for the direct sum of countably many infinite cyclic groups, which are here generated by the $\omega_p/\overline{\omega}_p$ with $p \equiv 1$ (mod 4) (cf. **[32]**). Since the group of similarity rotations of $\mathbb{Z}^2$ consists precisely of the set of $\mathbb{Z}^2$-directions (see Equation (1.2)), each nonzero element of $\mathrm{SOS}(\mathbb{Z}^2)$ is of the form $z/|z|$, $z \in \mathbb{Z}[i]$. Using unique factorisation in $\mathbb{Z}[i]$ again, we get

$$\frac{z}{|z|} = \left(\frac{1+i}{\sqrt{2}}\right)^k \prod_{p\equiv 1(4)} \left(\frac{\omega_p}{\sqrt{p}}\right)^{\ell_p},$$

where $0 \leqslant k < 8$ and $\ell_p \in \mathbb{Z}$ (other restrictions as in (1.4)). Since $(1+i)/\sqrt{2} = e^{\frac{2\pi i}{8}}$ is a primitive 8th root of unity, it generates the cyclic group C_8. Furthermore, one has indeed

$$\left(\frac{\omega_p}{\sqrt{p}}\right)^2 = \frac{\omega_p^2}{\omega_p \overline{\omega}_p} = \frac{\omega_p}{\overline{\omega}_p}.$$

This shows that the generators of $\mathrm{SOC}(\mathbb{Z}^2) = C_4 \times \mathbb{Z}^{(\aleph_0)}$ are the squares of the generators of $\mathrm{SOS}(\mathbb{Z}^2)$. Thus

$$\begin{aligned} \mathrm{SOC}(\mathbb{Z}^2) &= \left\{ x^2 \;\middle|\; x \in \mathrm{SOS}(\mathbb{Z}^2) \right\} \\ &=: \left(\mathrm{SOS}(\mathbb{Z}^2)\right)^2 . \end{aligned}$$

We find the structure of the factor group to be

$$\begin{aligned} \mathrm{SOS}(\mathbb{Z}^2)/\,\mathrm{SOC}(\mathbb{Z}^2) &\simeq (C_8/C_4) \times C_2^{(\aleph_0)} \\ &\simeq C_2 \times C_2^{(\aleph_0)}, \end{aligned}$$

where $C_2^{(\aleph_0)}$ stands for the direct sum of countably many cyclic groups of order 2. Hence, the factor group is the direct sum of cyclic groups of order 2, which means that it is an elementary Abelian 2-group.

To unfold the structure of the factor group $\mathrm{OS}(\mathit{\Gamma}) \,/\, \mathrm{OC}(\mathit{\Gamma})$ in general, we need the following result from the theory of Abelian groups.

PROPOSITION 1.22. [**41**, Thms. 5.1.9 and 5.1.12] *Let G be an Abelian group.*

(1) *If there is a prime number p such that $x^p = 1$ for all $x \in G$, then G is the direct sum of subgroups of order p.*
(2) *If there is a positive integer n such that $x^n = 1$ for all $x \in G$, then G is the direct sum of cyclic groups of prime power orders.* □

THEOREM 1.23. *Let $\mathit{\Gamma} \subset \mathbb{R}^d$ be a free S-module of rank d that spans $\mathbb{R}^d$. The group $\mathrm{OS}(\mathit{\Gamma}) \,/\, \mathrm{OC}(\mathit{\Gamma})$ is the direct sum of cyclic groups of prime power orders that divide d.*

PROOF. We consider again the group homomorphism $\eta\colon \mathrm{OS}(\mathit{\Gamma}) \longrightarrow \mathbb{R}^{\bullet}/K^{\bullet}$. Let $R \in \mathrm{OS}(\mathit{\Gamma})$. According to Lemma 1.15, one has $\alpha^d \in K^{\bullet}$ for any nonzero $\alpha \in \mathrm{scal}_{\mathit{\Gamma}}(R)$, which yields

$$\eta(R)^d = \mathrm{scal}_{\mathit{\Gamma}}(R)^d = (\alpha K^{\bullet})^d = \alpha^d K^{\bullet} = K^{\bullet} \tag{1.5}$$

in $\mathbb{R}^{\bullet}/K^{\bullet}$. Using the group isomorphism $\eta(\mathrm{OS}(\mathit{\Gamma})) \simeq \mathrm{OS}(\mathit{\Gamma}) \,/\, \mathrm{OC}(\mathit{\Gamma})$, this shows that the order of each element of $\mathrm{OS}(\mathit{\Gamma}) \,/\, \mathrm{OC}(\mathit{\Gamma})$ divides d. Proposition 1.22(2) then implies that the group $\mathrm{OS}(\mathit{\Gamma}) \,/\, \mathrm{OC}(\mathit{\Gamma})$ is the direct sum of cyclic groups of prime power orders. Consequently, the prime power order of each cyclic group divides d. □

EXAMPLE 1.24. Denote by $\{e_1, \dots , e_d\}$ the canonical basis of $\mathbb{R}^d$. Let $n \geqslant 1$ be a natural number with $\nu = \sqrt[d]{n} \notin \mathbb{Q}$. The $\mathbb{Z}$-span $\mathit{\Gamma}$ of $\{\, \nu^i e_i \mid 1 \leqslant i \leqslant d \,\}$ is a lattice in $\mathbb{R}^d$. Consider the cyclic permutation $\sigma = (12....d)$ of the symmetric group S_d. Then σ induces a linear isometry R of $\mathbb{R}^d$ by permuting the canonical basis vectors, i.e., $Re_i = e_{\sigma(i)}$. Since $\nu R\Gamma \subset \Gamma$, R is a similarity isometry of Γ. But R is not a coincidence isometry of Γ (because $\sqrt[d]{n} \notin \mathbb{Q}$). Setting $m = \min_i \{ \nu^i \mid \nu^i \in \mathbb{Q} \}$, one easily verifies that the factor group $\mathrm{OS}(\Gamma) \,/\, \mathrm{OC}(\Gamma)$ contains the cyclic group C_m of order m generated by the equivalence class of R. If m is not a prime power, then the fundamental theorem of finite Abelian groups states that C_m is the direct sum of cyclic groups of prime power orders. Examples of the module case can be constructed similarly.

COROLLARY 1.25. *Let $\Gamma \subset \mathbb{R}^d$ be a free S-module of rank d that spans $\mathbb{R}^d$. If $d = p$ is a prime number, then $\mathrm{OS}(\Gamma) \,/\, \mathrm{OC}(\Gamma)$ is an elementary Abelian p-group, i.e., it is the direct sum of cyclic groups of order p.* □

It seems that, in practice, the factor group is smaller in many cases; see the end of this section.

DEFINITION 1.26. Let $\Gamma \subset \mathbb{R}^d$ be a free S-module of rank d that spans $\mathbb{R}^d$ and denote by $\langle .\,, . \rangle$ the standard scalar product of $\mathbb{R}^d$. We call Γ an *S-module over K in $\mathbb{R}^d$* if it satisfies $\langle \gamma, \gamma \rangle \in K$ for all $\gamma \in \Gamma$.

EXAMPLE 1.27.
(1) Let $S = \mathbb{Z}$. Then, $K = \mathbb{Q}$ and the S-modules Γ over K in $\mathbb{R}^d$ are precisely the *rational lattices* in $\mathbb{R}^d$, i.e. those lattices having a basis such that the Gramian matrix has rational entries only; cf. [**17**] for examples.
(2) For $n \in \mathbb{N}$, S^n is an S-module over K in $\mathbb{R}^n$.
(3) Consider the quadratic number field $L := \mathbb{Q}(\tau)$, where τ is the golden ratio, i.e., $\tau = (1 + \sqrt{5})/2$. Then, the ring of integers $\mathcal{O}_L$ of L is $\mathbb{Z}[\tau]$. The icosian ring $\mathbb{I}$ is defined as follows,
$$\mathbb{I} = \langle (1,0,0,0), (0,1,0,0), \tfrac{1}{2}(1,1,1,1), \tfrac{1}{2}(1-\tau,\tau,0,1) \rangle_{\mathcal{O}_L} \subset L^4,$$
where $\langle \cdot \rangle_{\mathcal{O}_L}$ denotes the $\mathcal{O}_L$-span. $\mathbb{I}$ is an $\mathcal{O}_L$-module over L in $\mathbb{R}^4$ (see [**9**]). Further, both the standard body centred icosahedral module $\mathcal{M}_{\mathrm{B}}$ and the standard face centred icosahedral module $\mathcal{M}_{\mathrm{F}}$ of quasicrystallography are $\mathcal{O}_L$-modules over L in $\mathbb{R}^3$; cf. [**2, 10**] and references therein.
(4) Consider the quadratic number field $L := \mathbb{Q}(\sqrt{2})$. Then, $\mathcal{O}_L = \mathbb{Z}[\sqrt{2}]$ and further, the octahedral (or cubian) ring
$$\mathbb{K} = \langle (1,0,0,0), \tfrac{1}{\sqrt{2}}(1,1,0,0), \tfrac{1}{\sqrt{2}}(1,0,1,0), \tfrac{1}{2}(1,1,1,1) \rangle_{\mathcal{O}_L} \subset L^4$$
is an $\mathcal{O}_L$-module over L in $\mathbb{R}^4$; cf. [**9, 10**] and references therein.
(5) Consider the cyclotomic field $\mathbb{Q}(\zeta_m)$, where $m \geq 3$ and ζ_m is a primitive mth root of unity in $\mathbb{C}$ (e.g. $\zeta_m = e^{\frac{2\pi i}{m}}$). Recall that $\mathbb{Q}(\zeta_m)$ is a finite Galois extension of $\mathbb{Q}$ with maximal real subfield $L := \mathbb{Q}(\zeta_m) \cap \mathbb{R} = \mathbb{Q}(\zeta_m + \bar{\zeta}_m)$; cf. [**42**, Theorem 2.5]. Further, it is well known that $\mathcal{O}_{\mathbb{Q}(\zeta_m)} = \mathbb{Z}[\zeta_m]$ and $\mathcal{O}_L = \mathbb{Z}[\zeta_m + \bar{\zeta}_m]$; cf. [**42**, Theorem 2.6 and Proposition 2.16]. Moreover, since $\zeta_m^2 = (\zeta_m + \bar{\zeta}_m)\zeta_m - 1$, the ring $\mathbb{Z}[\zeta_m]$ is the $\mathbb{Z}[\zeta_m + \bar{\zeta}_m]$-span of the $\mathbb{R}$-basis $\{1, \zeta_m\}$ of $\mathbb{C}$. Identifying the

complex numbers $\mathbb{C}$ with $\mathbb{R}^2$, one can now verify that $\mathbb{Z}[\zeta_m]$ is an $\mathcal{O}_L$-module over L in $\mathbb{R}^2$. In particular, rings of integers in imaginary cyclotomic fields can be used to construct planar mathematical quasicrystals such as the vertex sets of Penrose, Ammann-Beenker or shield tilings; cf. [**8, 4, 6, 30**].

THEOREM 1.28. *For any S-module $\varGamma$ over K in $\mathbb{R}^d$ one has*

$$\mathrm{OS}(\varGamma)^2 \subset \mathrm{OC}(\varGamma),$$

where $\mathrm{OS}(\varGamma)^2 = \{\, R^2 \mid R \in \mathrm{OS}(\varGamma) \,\}$*. Thus, the factor group* $\mathrm{OS}(\varGamma) \,/\, \mathrm{OC}(\varGamma)$ *is an elementary Abelian 2-group, when d is even. If d is odd, one has* $\mathrm{OS}(\varGamma) = \mathrm{OC}(\varGamma)$.

PROOF. Let $R \in \mathrm{OS}(\varGamma)$. Then there exists an element $\alpha \in \mathbb{R}_+$ with $\alpha R \varGamma \subset \varGamma$. By assumption, one has $\langle \alpha R\gamma, \alpha R\gamma \rangle \in K$ for all $\gamma \in \varGamma$. Hence $\alpha^2 \in K^\bullet$, say $\alpha^2 = s_1/s_2$, where $s_1, s_2 \in S^\bullet$. Since $s_2\alpha^2 = s_1 \in S$ and $\alpha R\varGamma \subset \varGamma$, this yields

$$\varGamma \supset s_2 \alpha R(\alpha R \varGamma) = s_2 \alpha^2 R^2 \varGamma \subset R^2 \varGamma,$$

whence

$$s_1 R^2 \varGamma \subset (\varGamma \cap R^2 \varGamma).$$

Since $\varGamma$, $R^2\varGamma$ and $s_1 R^2 \varGamma$ are $\mathbb{Z}$-modules of the same finite rank, one obtains by Lemma 1.3 that both $[\varGamma : s_1 R^2 \varGamma]$ and $[R^2\varGamma : s_1 R^2 \varGamma]$ are finite. It follows that $\varGamma \sim R^2 \varGamma$, meaning that R^2 is a coincidence isometry of $\varGamma$. Consequently, $\mathrm{OS}(\varGamma)^2 \subset \mathrm{OC}(\varGamma)$. Thus, every element of the factor group $\mathrm{OS}(\varGamma) \,/\, \mathrm{OC}(\varGamma)$ is of order 1 or 2. By Proposition 1.22(1), the factor group is an elementary Abelian 2-group.

If d is odd, set $d = 2m + 1$ with $m \in \mathbb{N}$. By Lemma 1.15, one has

$$\alpha(\alpha^2)^m = \alpha^d \ \in K^\bullet.$$

This yields $\alpha \in K^\bullet$, because $\alpha^2 \in K^\bullet$. Thus $\eta(R) = \mathrm{scal}_\varGamma(R) = \alpha K^\bullet = K^\bullet$ in $\mathbb{R}^\bullet / K^\bullet$ for all $R \in \mathrm{OS}(\varGamma)$, whence $\mathrm{OS}(\varGamma) \,/\, \mathrm{OC}(\varGamma)$ is the trivial group. In other words, one has $\mathrm{OS}(\varGamma) = \mathrm{OC}(\varGamma)$ for d odd. □

EXAMPLE 1.29. Using the notation of Example 1.27(5), consider a cyclotomic field $\mathbb{Q}(\zeta_m)$ of class number one, meaning that its ring of integers $\mathbb{Z}[\zeta_m]$ is a unique factorisation domain; see [**32**] for more on this. $\mathbb{Z}[\zeta_m]$ is an $\mathcal{O}_L$-module over L in $\mathbb{R}^2$. Since we are working in 2-space, Theorem 1.28 implies that the factor group of similarity modulo coincidence isometries is an elementary Abelian 2-group. Combining the results of [**32**] and [**24**, Prop. 1.89], one immediately obtains

$$\mathrm{SOS}(\mathbb{Z}[\zeta_m]) / \, \mathrm{SOC}(\mathbb{Z}[\zeta_m]) \simeq C_2 \times C_2^{(\aleph_0)},$$

where SOS and SOC indicate the restriction to orientation-preserving isometries, and $C_2^{(\aleph_0)}$ stands for the direct sum of countably many cyclic groups of order 2.

Note that the proof of Theorem 1.23 concerning the structure of the factor group is non-constructive. In 2 dimensions, there are examples where the factor group actually is the direct sum of countably many cyclic groups as is the case for $\mathbb{Z}^2$ or in Example 1.29. Recent results [**44**] however suggest that the situation is simpler for odd dimensions d, namely that the factor group of any lattice $\Gamma \subset \mathbb{R}^d$ is the direct sum of only *finitely* many cyclic groups (of orders dividing d). The reason behind this is that the subgroup of squares $\{\, x^2 \mid x \in \mathrm{OS}(\Gamma)/\,\mathrm{OC}(\Gamma)\,\} \subset \mathrm{OS}(\Gamma)/\,\mathrm{OC}(\Gamma)$ seems to be finite in this situation.

Generally, the 2-dimensional case, where the groups of similarity and coincidence rotations are subgroups of $\mathrm{SO}(2,\mathbb{R})$, is rather well understood [**2, 32, 11**]. Since $\mathrm{SO}(3,\mathbb{R})$, in contrast to $\mathrm{SO}(2,\mathbb{R})$, is not Abelian, the situation gets more complicated and is far from being understood. As a first step towards broadening the understanding of the 3-dimensional case, the second chapter of this thesis is devoted to the simplest example in 3 dimensions, namely the cubic lattice $\mathbb{Z}^3$ and its group of coincidence isometries $\mathrm{O}(3,\mathbb{Q})$. An intriguing open problem is the classification of the finitely generated subgroups of $\mathrm{O}(3,\mathbb{Q})$. Whereas all finite subgroups of $\mathrm{O}(3,\mathbb{Q})$ have been known for a very long time, even the next step of determining the 2-generator subgroups has not been accomplished. In what follows, we shall classify those 2-generator subgroups of $\mathrm{SO}(3,\mathbb{Q})$ that are generated by two rotations of finite order whose rotation axes also enclose an angle of finite order. These groups appear in the theory of tilings of $\mathbb{R}^3$, e.g. as orientation group of the "quaquaversal tiling"; see Remark 2.32.

CHAPTER 2

On 2-Generator Subgroups of $\mathrm{SO}(3, \mathbb{Q})$

In this chapter, we maintain a rather group theoretical view concerning coincidence isometries in 3-space [**2, 10, 23**]. More precisely, we investigate the orthogonal group $\mathrm{O}(3, \mathbb{Q})$, which is the group of coincidence isometries of $\mathbb{Z}^3$ (cf. [**2**]). It is of basic interest and therefore presents itself as a natural starting point for investigating the group of coincidence isometries in three dimensions. Since $\mathbb{Z}^3$ is a rational lattice (see Example 1.27(1)), Theorem 1.28 implies that the groups of similarity and coincidence isometries coincide, i.e., $\mathrm{OS}(\mathbb{Z}^3) = \mathrm{OC}(\mathbb{Z}^3)$. First, we recall some algebraic and group theoretic results.

2.1. Basic algebraic tools

2.1.1. Cyclotomic fields. Let $n \in \mathbb{N}$ and let ζ_n be a fixed primitive nth root of unity in $\mathbb{C}$, e.g. $\zeta_n = e^{\frac{2\pi i}{n}}$. Then $\mathbb{Q}(\zeta_n)$ is the corresponding cyclotomic field. If m and n are relatively prime natural numbers, then $\mathbb{Q}(\zeta_m) = \mathbb{Q}(\zeta_m^n)$, because ζ_m^n is also a primitive m-th root of unity. If $\bar{\cdot}$ denotes complex conjugation, then $\mathbb{Q}(\zeta_n + \overline{\zeta_n}) = \mathbb{Q}(2\cos(2\pi/n))$ is the maximal real subfield of $\mathbb{Q}(\zeta_n)$; cf. [**42**]. Furthermore, let φ denote Euler's totient function, which is defined for all $n \in \mathbb{N}$ as follows.

$$\varphi(n) = \mathrm{card}\left(\left\{k \in \mathbb{N} \mid 1 \leq k \leq n \text{ and } \gcd(k, n) = 1\right\}\right)$$

For any field extension F/K denote by $[F : K]$ the degree of the field extension.

LEMMA 2.1. [**27**, Ch. V, Prop. 1.2] *Let $E/F/K$ be an extension of fields. Then one has $[E : K] = [E : F] \cdot [F : K]$.* □

THEOREM 2.2. [**42**, Thm. 2.5] *The field extension $\mathbb{Q}(\zeta_n)/\mathbb{Q}$ is a Galois extension of degree $\varphi(n)$ with Abelian Galois group $(\mathbb{Z}/n\mathbb{Z})^\times$.* □

COROLLARY 2.3. *If $n \geqslant 3$, one has $[\mathbb{Q}(\zeta_n + \overline{\zeta_n}) : \mathbb{Q}] = \varphi(n)/2$.*

PROOF. This follows from Theorem 2.2 using the degree formula and Galois theory. Namely, $\mathbb{Q}(\zeta_n + \overline{\zeta_n})$ is the fixed field of complex conjugation and therefore $\mathbb{Q}(\zeta_n)/\mathbb{Q}(\zeta_n + \overline{\zeta_n})$ is of degree 2. □

THEOREM 2.4. [**27**, Ch. VI, Cor. 3.2] *If $m, n \in \mathbb{N}$ with $\gcd(m, n) = 1$, then*

$$\mathbb{Q}(\zeta_m) \cap \mathbb{Q}(\zeta_n) = \mathbb{Q}.$$

□

COROLLARY 2.5. *If* $m, n \in \mathbb{N}$ *with* $\gcd(m,n) = 1$*, one has*

$$\mathbb{Q}\Big(2\cos\Big(\frac{2\pi}{m}\Big)\Big) \cap \mathbb{Q}\Big(2\cos\Big(\frac{2\pi}{n}\Big)\Big) = \mathbb{Q}.$$

PROOF. This follows immediately from Theorem 2.4, since $\mathbb{Q}(2\cos(2\pi/t)) = \mathbb{Q}(\zeta_t) \cap \mathbb{R}$ for any $t \in \mathbb{N}$. □

2.1.2. Group presentations of some well-known groups. A group F is called *free* on a subset Y of F, if for any group H and any map $\theta\colon Y \longrightarrow H$, there is a unique group homomorphism $\theta'\colon F \longrightarrow H$ such that the following diagram commutes.

$$\begin{tikzcd} Y \arrow[d, hook] \arrow[dr, "\theta"] & \\ F \arrow[r, dashed, "\exists!\theta'"'] & H \end{tikzcd}$$

The cardinality of Y is called the *rank of* F; see [**25**] for more on this.

A common example of a free group is the group of integers $\mathbb{Z}$, which is free on $\{1\} \subset \mathbb{Z}$ and therefore has rank 1. On the contrary, all nontrivial finite groups fail to be free.

REMARK 2.6. One can show that every element h of a free group F on Y can uniquely be written in the form

$$h = y_1^{r_1} y_2^{r_2} \dots y_s^{r_s}, \tag{2.1}$$

where $s \geqslant 0$, $r_i \neq 0$ and $y_i \neq y_{i+1}$. Here, $s = 0$ is interpreted as $h = e$, the identity element of F. On the other hand, if G is a group, $Y \subset G$ and every element has a unique expression of the form (2.1), then G is free on X; cf. [**35**, Ch. 2.1].

Let G be a group and F be a free group on some subset Y. If $\pi\colon F \longrightarrow G$ is an epimorphism, then $G \simeq F/\operatorname{Ker}(\pi)$. Let $R \subset F$ such that the normal closure of R in F is $\operatorname{Ker}(\pi)$. With this notation write

$$G = \langle Y \mid R \rangle \tag{2.2}$$

and call the right hand side a *presentation of* G. The elements of Y are called *generators* while those of R are called *defining relators*. G is called *finitely presented*, if it has a presentation of the form (2.2), where Y and R are finite sets. Instead of listing Y and R, one often lists the generators $X := \pi(Y)$ of G and the defining relations in these generators $x \in X$. By abuse of language, this reads in symbols

$$G = \langle X \mid r(x), r \in R \rangle.$$

THEOREM 2.7. [**25**, Ch. 2, Thm. 1] *Every group has a presentation and every finite group is finitely presented.* □

EXAMPLE 2.8. (1) A presentation of the cyclic group C_t of order $t \in \mathbb{N}$ is given by $C_t = \langle x \mid x^t \rangle$.

(2) The dihedral group D_m of order $2m$ has a presentation $D_m = \langle y, z \,|\, y^m, z^2, (yz)^2\rangle$, cf. [**25**, Ch. 4, Thm. 1].

Different presentations may describe the same group and it is therefore of interest to determine when this is the case. We shall write $\langle X \,|\, R\rangle = \langle X' \,|\, R'\rangle$, if two presentations $\langle X \,|\, R\rangle$ and $\langle X' \,|\, R'\rangle$ present the same group. Given a presentation of a group, there are four so-called Tietze transformations which correspond to adding and removing a redundant relator, and adding and removing a redundant generator.

THEOREM 2.9. [**25**, Ch. 5, Thms. 1 and 2] *Two presentations $\langle X \,|\, R\rangle$ and $\langle X' \,|\, R'\rangle$ are presentations of the same group (up to isomorphism) if and only if one can be transformed into the other by means of a finite sequence of Tietze transformations.* □

The symmetry group on four elements S_4 is generated by the permutations $t = (12)$ and $s = (234)$. One can show (cf. [**19**, Ch. 1.5]) that a group presentation is given by

$$S_4 = \langle s, t \,|\, s^3, t^2, (st)^4\rangle. \tag{2.3}$$

LEMMA 2.10. *Another possible presentation for the symmetric group S_4 is*

$$S_4 = \langle u, v \,|\, u^4, v^4, (u^2v)^2, (uv^2)^2, (uv)^3\rangle.$$

PROOF. Due to Theorem 2.9, it suffices to show that the presentation (2.3) can be transformed into the one above by a finite sequence of Tietze transformations, i.e., by adding and removing superfluous generators and relators. We start by adding the redundant generators $u = s^{-1}t$ and $v = st$, leading to the additional relator v^4. From $u = tv^{-1}t$ we deduce $u^4 = tv^{-4}t = e$, the identity element. In the next step, we add the relator u^4 and remove the relator $(st)^4$. Thus, one has

$$\begin{aligned}\langle s, t \,|\, s^3, t^2, (st)^4\rangle &= \langle s, t, u, v \,|\, s^3, t^2, (st)^4, uts, vts^{-1}, v^4\rangle\\ &= \langle s, t, u, v \,|\, s^3, t^2, uts, vts^{-1}, v^4, u^4\rangle\end{aligned}$$

From $u^4 = e$, one finds $s^{-1}ts^{-1}t = u^2 = u^{-2} = tsts$. Hence

$$\big(u^2v\big)^2 = \big((s^{-1}ts^{-1}t)(st)\big)^2 = \big((tsts)(st)\big)^2 = \big(tsts^2t\big)^2 = e\,.$$

Also, one has

$$\big(uv^2\big)^2 = \big(s^{-1}(tsts)t\big)^2 = \big(s^{-1}\left(s^{-1}ts^{-1}t\right)t\big)^2 = \big(s^{-2}ts^{-1}\big)^2 = e\,.$$

In order to remove the generator t we express it as $t = su$ and replace t accordingly in all relators involving t. This yields

$$\left\langle s, t \,\middle|\, s^3, t^2, (st)^4\right\rangle = \left\langle s, u, v \,\middle|\, s^3, (su)^2, vu^{-1}s, v^4, u^4, \big(u^2v\big)^2, \big(uv^2\big)^2\right\rangle.$$

We proceed similarly with the generator s. It can be expressed as $s = uv^{-1}$. Hence the presentation above is transformed into

$$\left\langle u, v \,\middle|\, \big(uv^{-1}\big)^3, \big(uv^{-1}u\big)^2, v^4, u^4, \big(u^2v\big)^2, \big(uv^2\big)^2\right\rangle.$$

Using further Tietze transformations by adding and removing redundant relators, one eventually arrives at

$$\left\langle u, v \,\middle|\, \left(uv^{-1}\right)^3, \left(uv^{-1}u\right)^2, v^4, u^4, \left(u^2v\right)^2, \left(uv^2\right)^2 \right\rangle = \langle u, v \mid u^4, v^4, (u^2v)^2, (uv^2)^2, (uv)^3 \rangle.$$

□

2.1.3. Free products of groups. Let $\{ G_\lambda \mid \lambda \in \Lambda \}$ be a nonempty set of groups. A group G together with group homomorphisms $\iota_\lambda \colon G_\lambda \longrightarrow G$ is called a *free product of the* G_λ*'s* if it satisfies the following universal property.

Given any group C and homomorphisms $\psi_\lambda \colon G_\lambda \longrightarrow C$, there is a unique homomorphism $\psi \colon G \longrightarrow C$ such that the diagram below commutes for all $\lambda \in \Lambda$.

$$\begin{tikzcd} G_\lambda \arrow[d, "\iota_\lambda"'] \arrow[dr, "\psi_\lambda"] & \\ G \arrow[r, dashed, "\exists!\psi"'] & C \end{tikzcd}$$

One easily deduces that ι_λ is injective for all $\lambda \in \Lambda$ by taking $C = G_\mu$, $\mu \in \Lambda$, and ψ_λ to be the identity function if $\lambda = \mu$, and $\psi_\lambda \equiv e$ otherwise, where e is the identity element of G_μ.

A free product exists for any nonempty family of groups $\{ G_\lambda \mid \lambda \in \Lambda \}$ and it is unique up to isomorphism (cf. [**36**, Thms. 11.50 and 11.51]). Therefore, we may speak of the free product of the G_λ's and denote it by

$$\mathop{\star}_{\lambda \in \Lambda} G_\lambda,$$

or by

$$G_{\lambda_1} \star G_{\lambda_2} \star \ldots \star G_{\lambda_r},$$

if $\Lambda = \{\lambda_1, \lambda_2, \ldots, \lambda_r\}$ is finite.

Example 2.11. (1) If F_1 and F_2 are free groups of rank r_1 and r_2, respectively, then $F_1 \star F_2$ is a free group of rank $r_1 + r_2$.
(2) A free group F is a free product of infinite cyclic groups.

Remark 2.12. If $G = \mathop{\star}_{\lambda \in \Lambda} G_\lambda$ is the free product of the groups G_λ, $\lambda \in \Lambda$, then every element $g \in G$ can be uniquely written in the form

$$g = g_1 g_2 \ldots g_\ell \quad (\ell \geqslant 0), \tag{2.4}$$

where $e \neq g_i \in G_{\lambda_i}$ and $\lambda_i \neq \lambda_{i+1}$ for $1 \leqslant i \leqslant \ell - 1$. Here, $\ell = 0$ is understood as $g = e$, the identity element of G. This is called the *normal form of* g. Note that the product of two elements $g, h \in G$ is given by juxtaposition; cf. [**35**, Ch. 6.2] for details.

In turn, the free product of groups is characterised by the existence of a normal form. Namely, if G is a group generated by subgroups G_λ, $\lambda \in \Lambda$, and every element has a unique expression of the form (2.4), then G is the free product of the G_λ's (cf. [**35**, Thm. 6.2.4]).

Consider two groups G_1 and G_2 and their free product $G_1 \star G_2$ with monomorphisms $\iota_i \colon G_i \longrightarrow G_1 \star G_2$ for $i \in \{1, 2\}$. Now let H be a group that is isomorphic to subgroups of G_1 and G_2 via monomorphisms $\varphi_i \colon H \longrightarrow G_i$, ($i \in \{1, 2\}$). Furthermore let N be the normal closure in $G_1 \star G_2$ of the set

$$\{\, \varphi_1(h)\,(\varphi_2(h))^{-1} \mid h \in H \,\}.$$

Then the factor group

$$G = (G_1 \star G_2)\,/N$$

is called the *free product of* G_1 *and* G_2 *with amalgamated subgroup* H *(with respect to* φ_1 *and* φ_2*)*. Note that G depends on the choice of the monomorphisms φ_1 and φ_2. Nevertheless, we denote G by $G_1 \star_H G_2$.

Since $\varphi_1(h) \equiv \varphi_2(h) \bmod N$ for all $h \in H$, the subgroups $(\varphi_1(H)N)/N$ and $(\varphi_2(H)N)/N$ are identified in $G_1 \star_H G_2$.

Theorem 2.13. [**36**, Thm. 11.58] *$G_1 \star_H G_2$ satisfies the following universal property. Let $\psi_i \colon G_i \longrightarrow T$ $(i \in \{1, 2\})$ be homomorphisms into some group T with $\psi_1|_H = \psi_2|_H$. Then there exists a unique homomorphism $\psi \colon G_1 \star_H G_2 \longrightarrow T$ that makes the diagram below commutative.*

$$\begin{tikzcd}
H \arrow[r, hook, "\varphi_1"] \arrow[d, hook, "\varphi_2"'] & G_1 \arrow[d, "\tau_1"] \arrow[ddr, "\psi_1"] & \\
G_2 \arrow[r, "\tau_2"] \arrow[drr, "\psi_2"'] & G_1 \star_H G_2 \arrow[dr, dashed, "\exists!\psi"] & \\
& & T
\end{tikzcd}$$

Here, $\tau_i := \mathrm{pr} \circ \iota_i$, where pr *is the canonical projection from $G_1 \star G_2$ onto $G_1 \star_H G_2$.*

In other words, $G_1 \star_H G_2$ together with τ_1 and τ_2 is the pushout of $\varphi_1 \colon H \longrightarrow G_1$ and $\varphi_2 \colon H \longrightarrow G_2$ in the category of groups. □

Corollary 2.14. *If G' is any other group with group homomorphisms $\tau_i' \colon G_i \longrightarrow G'$ $(i \in \{1, 2\})$ satisfying the universal property of Theorem* 2.13*, then $G' \simeq G_1 \star_H G_2$. Thus, the universal property characterises free products with amalgamated subgroup.* □

Taking a closer look at the involved homomorphisms in the diagram above, one finds that the homomorphisms $\tau_i \colon G_i \longrightarrow G_1 \star_H G_2$ are injective. Thus, one can identify G_i with its image $\tau_i(G_i)$ making G_i a subgroup of $G_1 \star_H G_2$. Moreover, $G_1 \star_H G_2$ is generated by

$\tau_1(G_1)$ and $\tau_2(G_2)$, and their intersection is $\tau_1(H) = \tau_2(H)$, the latter being isomorphic to H; cf. [**36**, Thm. 10.67] for details. Therefore H can as well be viewed as a subgroup of $G_1 \star_H G_2$.

THEOREM 2.15. [**36**, Thm. 11.58] *If G_1 has a presentation $\langle Y_1 \,|\, R_1\rangle$ and G_2 has a presentation $\langle Y_2 \,|\, R_2\rangle$ such that $Y_1 \cap Y_2 = \varnothing$, then the free product of G_1 and G_2 with amalgamated subgroup H has a presentation*

$$\Big\langle\, Y_1 \cup Y_2 \;\Big|\; R_1 \cup R_2 \cup \{\, \varphi_1(h)\,(\varphi_2(h))^{-1} \mid h \in H\,\}\,\Big\rangle.$$

□

EXAMPLE 2.16. (1) Let $C_4 = \langle x \,|\, x^4\rangle$ and $C_6 = \langle y \,|\, y^6\rangle$ be cyclic groups of order 4 and 6 respectively. Since x^2 and y^3 both have order 2, C_4 and C_6 have a common subgroup C_2. More precisely, let $C_2 = \langle z \,|\, z^2\rangle$. The homomorphisms $\varphi_1\colon C_2 \longrightarrow C_4$ defined by $\varphi_1(z) = x^2$ and $\varphi_2\colon C_2 \longrightarrow C_6$ defined by $\varphi_2(z) = y^3$ are injective and the subgroups $\varphi_1(C_2)$ of C_4 and $\varphi_2(C_2)$ of C_6 are isomorphic. By Theorem 2.15, the free product of C_4 and C_6 with amalgamated subgroup C_2 has a presentation

$$\begin{aligned} C_4 \star_{C_2} C_6 &= \big\langle x, y \,\big|\, x^4, y^6, \{\, \varphi_1(h)\,(\varphi(h))^{-1} \mid h \in H\,\}\big\rangle \\ &= \big\langle x, y \,\big|\, x^4, y^6, \varphi_1(z)\,(\varphi_2(z))^{-1}\,\big\rangle \\ &= \big\langle x, y \,\big|\, x^4, y^6, x^2y^{-3}\big\rangle \\ &= \big\langle x, y \,\big|\, x^4, x^2y^{-3}\big\rangle. \end{aligned}$$

Thus x^2 and y^3 are identified in $C_4 \star_{C_2} C_6$.

(2) The dihedral group D_2 of order 4 has a presentation $\langle \alpha, \beta \,|\, \alpha^2, \beta^2, (\alpha\beta)^2\rangle$. It obviously contains $C_2 = \langle \beta \,|\, \beta^2\rangle$ as a subgroup. Considering another copy of $D_2 = \langle \gamma, \delta \,|\, \gamma^2, \delta^2, (\gamma\delta)^2\rangle$ such that the sets of generators are disjoint, we can form the free product of two copies of D_2 with amalgamated subgroup C_2. By Theorem 2.15, it has a presentation

$$\begin{aligned} D_2 \star_{C_2} D_2 &= \big\langle \alpha, \beta, \gamma, \delta \,\big|\, \alpha^2, \beta^2, (\alpha\beta)^2, \gamma^2, \delta^2, (\gamma\delta)^2, \beta\delta^{-1}\big\rangle \\ &= \big\langle \alpha, \beta, \gamma \,\big|\, \alpha^2, \beta^2, (\alpha\beta)^2, \gamma^2, (\gamma\beta)^2\big\rangle. \end{aligned}$$

This amounts to identifying β and δ.

Hence it is easily possible to achieve a presentation of the free product $G_1 \star_H G_2$ with amalgamated subgroup H from presentations of G_1 and G_2. Given a presentation, it may be very difficult though (if not impossible) to determine the order of the group being presented; cf. [**36**, Ch. 12]. Without taking a presentation into account, there is a useful theorem to determine whether a free product with amalgamated subgroup has infinite order.

THEOREM 2.17. [**35**, Thm. 6.4.3] *If both G_1 and G_2 are different from H, then $G_1 \star_H G_2$ has an element of infinite order. Moreover, any element of finite order is conjugate to an element of G_1 or G_2.*

Thus, both groups $C_4 \star_{C_2} C_6$ and $D_2 \star_{C_2} D_2$ of Example 2.16 are infinite. Next, we consider the case where one of the G_i's coincides with the amalgamated subgroup H.

LEMMA 2.18. *One has $H \star_H G_2 \simeq G_2$.*

PROOF. By Corollary 2.14, it suffices to show that G_2 satisfies the universal property of $H \star_H G_2$. Let $\psi_1 \colon H \longrightarrow T$ and $\psi_2 \colon G_2 \longrightarrow T$ be homomorphisms into some group T with $\psi_1 = \psi_2|_H$. Consider the following diagram.

$$\begin{tikzcd}
H \arrow[r, hook, "id_H"] \arrow[d, hook, "\varphi_2"'] & H \arrow[d, "\varphi_2"'] \arrow[ddr, "\psi_1"] & \\
G_2 \arrow[r, "id_{G_2}"] \arrow[drr, "\psi_2"'] & G_2 \arrow[dr, dashed, "\exists!\psi"] & \\
 & & T
\end{tikzcd}$$

We have to show that there is exactly one homomorphism $\psi \colon G_2 \longrightarrow T$ making the diagram commute. Set $\psi = \psi_2$. Then one has $\psi \circ \mathrm{id}_{G_2} = \psi_2$ and $\psi \circ \varphi_2 = \psi_2 \circ \varphi_2 = \psi_1$ by assumption. Furthermore ψ is uniquely determined by $\psi \circ \mathrm{id}_{G_2} = \psi_2$. $\square$

2.2. $\mathrm{O}(3,\mathbb{Q})$ as a subgroup of $\mathrm{O}(3,\mathbb{R})$

The orthogonal group $\mathrm{O}(3,\mathbb{Q})$ is at the centre of our interest. It is itself a subgroup of $\mathrm{O}(3,\mathbb{R})$. Moreover, it is even dense in $\mathrm{O}(3,\mathbb{R})$ as we shall see now. A subset $\Lambda \subset \mathbb{R}^d$ with $d \in \mathbb{N}$ is called *relatively dense*, if there is a radius $R > 0$ such that every ball $B_R(x)$ with radius R around $x \in \mathbb{R}^d$ contains at least one element of Λ. An element u of the unit $(d-1)$-sphere $\mathbb{S}^{d-1}$ is called a Λ-*direction*, if it is a scalar multiple of a nonzero element of the difference set $\Lambda - \Lambda$.

LEMMA 2.19. [**24**, Lemma 1.59] *Let $d \geqslant 2$ and let $\Lambda \subset \mathbb{R}^d$ be relatively dense. Then, the set of Λ-directions is dense in $\mathbb{S}^{d-1}$.* $\square$

THEOREM 2.20. *Let $d \geqslant 2$. Then $\mathrm{O}(d,\mathbb{Q})$ is dense in $\mathrm{O}(d,\mathbb{R})$.*

PROOF. Let $T \in \mathrm{O}(d,\mathbb{R})$ and $\varepsilon > 0$. It suffices to show that there exists $N \in \mathrm{O}(d,\mathbb{Q})$ such that $\|T - N\|_\infty < \varepsilon$, where $\|\cdot\|_\infty$ denotes the maximum norm on $\mathbb{R}^{d^2}$. By the theorem of Cartan-Dieudonné (cf. [**29**, Thm. 43:3]), T is a composition of at most d reflections in hyperplanes through the origin. Denote by r the number of reflections in the composition. For $i \leqslant r$ these reflections are given by

$$\begin{aligned} \sigma_{u_i} : \mathbb{R}^d &\longrightarrow \mathbb{R}^d, \\ x &\mapsto x - \frac{2\langle x, u_i\rangle}{\langle u_i, u_i\rangle}\, u_i, \end{aligned}$$

where $u_i \in \mathbb{S}^{d-1}$ is orthogonal to the respective hyperplane. Writing $u_i = (u_{i1}, u_{i2}, \dots, u_{id})^t$, the representing matrix M_i of σ_{u_i} with respect to the canonical basis $\{e_1, \dots, e_d\}$ of $\mathbb{R}^d$ reads

$$M_i = (m_{kj}), \quad \text{where } m_{kj} = \begin{cases} 1 - 2u_{ik}^2, & k = j \\ -2u_{ik}u_{ij}, & \text{else.} \end{cases} \tag{2.5}$$

Since $\mathbb{Z}^d \subset \mathbb{R}^d$ is relatively dense, the set of $\mathbb{Z}^d$-directions $\{\, x/\|x\| \mid x \in \mathbb{Z}^d \,\}$ is dense in $\mathbb{S}^{d-1}$ by Lemma 2.19. Thus, for every $\delta > 0$ and every u_i, there is a $\mathbb{Z}^d$-direction v_i such that $\|u_i - v_i\|_\infty < \delta$, where $\|\cdot\|_\infty$ denotes also the maximum norm on $\mathbb{R}^d$. Denoting by N_i the representing matrix (with respect to the canonical basis) of the reflection σ_{v_i} in the hyperplane orthogonal to v_i, one deduces from (2.5) that $M_i - N_i = (b_{kj})$ with $b_{kj} = 2(v_{ik}v_{ij} - u_{ik}u_{ij})$. Therefore, one has

$$\|M_i - N_i\|_\infty = \max_{k,j}\{2 \cdot |v_{ik}v_{ij} - u_{ik}u_{ij}|\}.$$

We take a closer look at the matrix entries and use $u_i, v_i \in \mathbb{S}^{d-1}$ to find

$$\begin{aligned} 2 \cdot |v_{ik}v_{ij} - u_{ik}u_{ij}| &= 2 \cdot |v_{ik}v_{ij} - u_{ik}v_{ij} + u_{ik}v_{ij} - u_{ik}u_{ij}| \\ &\leqslant 2 \cdot |v_{ik} - u_{ik}| \cdot |v_{ij}| + 2 \cdot |u_{ik}| \cdot |v_{ij} - u_{ij}| \\ &\leqslant 2 \cdot |v_{ik} - u_{ik}| + 2 \cdot |v_{ij} - u_{ij}| \\ &\leqslant 4 \cdot \|u_i - v_i\|_\infty < 4\delta. \end{aligned}$$

Hence, one has $\|M_i - N_i\|_\infty < 4\delta$ for all $i \leqslant r$.

Let $z_i \in \mathbb{Z}^d$ such that $v_i = z_i/\|z_i\|$. Obviously, one has $\sigma_{v_i} = \sigma_{z_i}$ and the entries of the representing matrix N_i are elements of $\mathbb{Q}$, because

$$\sigma_{z_i}(e_t) = e_t - \frac{2\langle e_t, z_i\rangle}{\langle z_i, z_i\rangle}\, z_i \in \mathbb{Q}^d$$

for all $1 \leqslant t \leqslant d$. Thus $N_i \in \mathrm{O}(d, \mathbb{Q})$ for all i, which yields $N := N_1N_2 \dots N_r \in \mathrm{O}(d, \mathbb{Q})$. Finally, we deduce from the fact that matrix multiplication is a continuous function that

$$\|T - N\|_\infty = \|M_1 \dots M_r - N_1 \dots N_r\|_\infty < \varepsilon$$

for a suitable $\delta > 0$. □

Corollary 2.21. *Let $d \geqslant 2$. Then* $\mathrm{SO}(d, \mathbb{Q})$ *is dense in* $\mathrm{SO}(d, \mathbb{R})$.

The special orthogonal group $\mathrm{SO}(d,\mathbb{R})$ is a normal subgroup of $\mathrm{O}(d,\mathbb{R})$ of index 2. If d is even, then every element of $\mathrm{O}(d,\mathbb{R})$ can uniquely be written as the product of a rotation and an element of $C_2 = \{S, E_d\}$, where E_d is the identity matrix and S is the diagonal matrix

$$S = \begin{pmatrix} -1 & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{pmatrix}.$$

For if $M \in \mathrm{O}(d,\mathbb{R}) \setminus \mathrm{SO}(d,\mathbb{R})$, one can write $M = (MS)S$. This representation is unique due to $\mathrm{SO}(d,\mathbb{R}) \cap \{S, E_d\} = \{E_d\}$. Hence $\mathrm{O}(d,\mathbb{R})$ is the semidirect product of $\mathrm{SO}(d,\mathbb{R})$ by $C_2 = \{S, E_d\}$, denoted by $\mathrm{O}(d,\mathbb{R}) = \mathrm{SO}(d,\mathbb{R}) \rtimes \{S, E_d\} \simeq \mathrm{SO}(d,\mathbb{R}) \rtimes C_2$.

If d is odd, then one easily sees that $\mathrm{O}(d,\mathbb{R})$ is the direct product

$$\mathrm{O}(d,\mathbb{R}) = \mathrm{SO}(d,\mathbb{R}) \times \{\pm E_d\} \simeq \mathrm{SO}(d,\mathbb{R}) \times C_2.$$

The analogous result holds for the orthogonal group of degree 3 over $\mathbb{Q}$, namely

$$\mathrm{O}(3,\mathbb{Q}) = \mathrm{SO}(3,\mathbb{Q}) \times C_2.$$

From now on, we can therefore restrict our attention to the special orthogonal group $\mathrm{SO}(3,\mathbb{Q})$, which is the group of coincidence rotations of $\mathbb{Z}^3$.

Studying the structure of $\mathrm{SO}(3,\mathbb{Q})$ leads to the task of determining its subgroups. The classification of finite subgroups is well known (cf. Section 2.3). What about other types of subgroups, e.g. finitely generated ones? In a first step towards an answer to this question, we shall consider 2-generator subgroups. More precisely, our aim is to classify special 2-generator subgroups: the so-called generalised dihedral groups (cf. Section 2.4).

2.3. Finite subgroups of $\mathrm{SO}(3,\mathbb{Q})$

In this section, the well-known classification of finite subgroups of the special orthogonal group $\mathrm{SO}(3,\mathbb{R})$ is reviewed [**18, 34, 19**], and the classification of finite subgroups of $\mathrm{SO}(3,\mathbb{Q})$ is derived. We start by considering the rotation symmetry groups of the five Platonic solids, i.e. the groups of orientation-preserving isometries that leave the Platonic solid invariant.

The tetrahedron has 4 vertices, 6 edges and its 4 faces are equilateral triangles. At each vertex exactly 3 of its faces meet, see Figure 2.1. The rotation symmetry group $\mathcal{T}$ of the tetrahedron consists of the following linear isometries.

- the identity
- the rotation by an angle of $2\pi/3$ or $4\pi/3$ about one of the 4 axes through a vertex and the centre of the opposite face
- the rotation by π about one of the 3 axes through the midpoints of two opposite edges

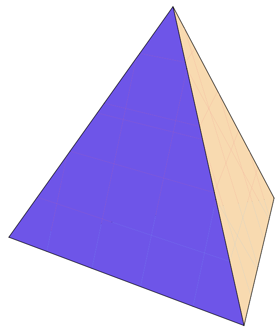

FIGURE 2.1. Tetrahedron

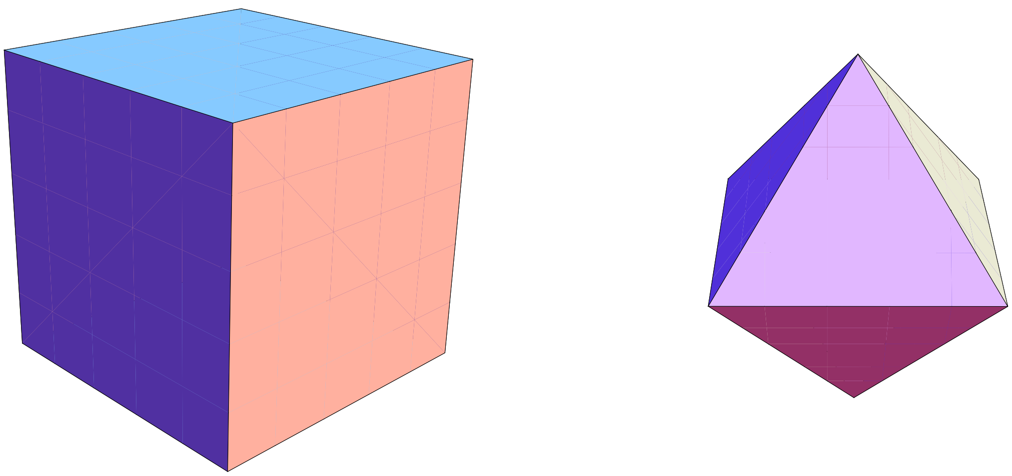

FIGURE 2.2. Cube and octahedron

$\mathcal{T}$ thus has 12 elements and one observes, by considering the permutations of the 4 vertices induced by the rotations, that $\mathcal{T}$ is isomorphic to the alternating group A_4 (cf. [**34**, Ch. 1]). Therefore, a presentation of $\mathcal{T}$ in terms of generators and defining relators is given by $\mathcal{T} = \langle s, u \mid s^3, u^2, (su)^3 \rangle$.

The cube (or hexahedron) consists of 6 square faces with 3 meeting at each of the 8 vertices. The dual polyhedron of the cube is obtained by taking the centres of the faces to be the vertices of the dual polyhedron whereas the edges of the dual are obtained by connecting the centres of adjacent faces of the cube. It is called octahedron. Thus the number of faces of the cube is the number of vertices of the dual and vice versa. One easily observes that a polyhedron and its dual have the same symmetry group. Note that the dual of the tetrahedron is again a tetrahedron.

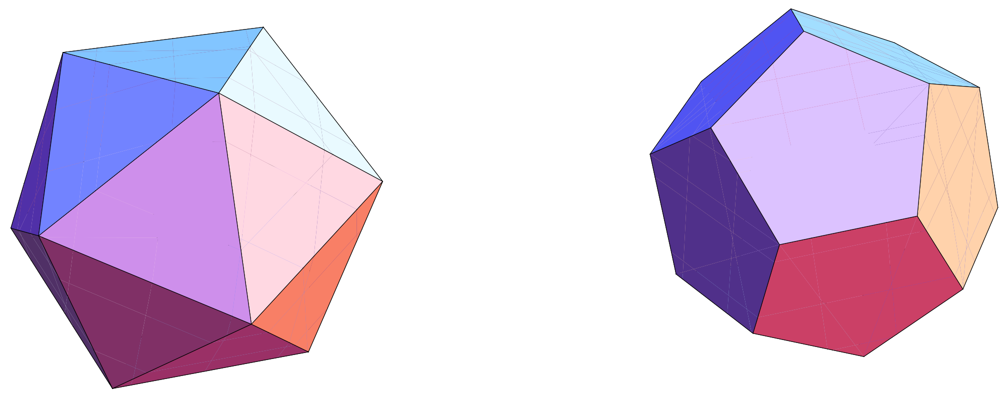

FIGURE 2.3. Icosahedron and dodecahedron

The elements of the rotation symmetry group $\mathcal{O}$ of the cube and of the octahedron, respectively, are

- the identity
- the rotation about one of the 4 diagonals of the cube by an angle of $2\pi/3$ or $4\pi/3$
- the rotation about one of the 3 axes through the centres of opposite faces of the cube by $\pi/2$, π or $3\pi/2$
- the rotation by π about one of the 6 axes through the midpoints of opposite edges of the cube

$\mathcal{O}$ has 24 elements and it permutes the four diagonals of the cube. Hence $\mathcal{O}$ is isomorphic to the symmetric group S_4 and has the presentation $\mathcal{O} = \langle s, t \mid s^3, t^2, (st)^4 \rangle$.

The last two remaining Platonic solids – the icosahedron and the dodecahedron – form another pair of dual polyhedra. The icosahedron has 12 vertices, 30 edges and its 20 faces are identical equilateral triangles, five of which meet at every vertex. The set of vertices can be subdivided (in five different ways) to form 3 concentric orthogonal rectangles whose side lengths are in the golden ratio $\tau = (1 + \sqrt{5})/2$, see Figure 2.4 for an illustration. The rotation symmetry group $\mathcal{I}$ of the icosahedron and of the dodecahedron, respectively, is comprised of

- the identity
- the rotation about one of the 6 axes through opposite vertices of the icosahedron by an angle of $2\pi/5$, $4\pi/5$, $6\pi/5$ or $8\pi/5$
- the rotation by $2\pi/3$ or $4\pi/3$ about one of the 10 axes through centres of opposite faces of the icosahedron

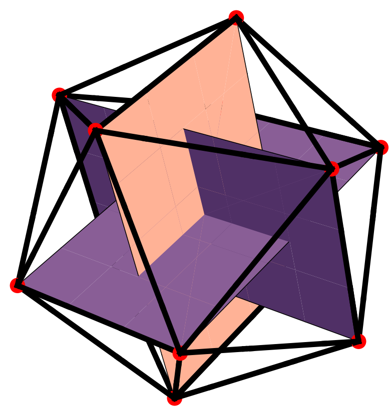

FIGURE 2.4. Icosahedron with golden rectangles

- the rotation about one of the 15 axes through the midpoints of opposite edges of the icosahedron by π

Hence $\mathcal{I}$ has 60 elements and one verifies that it is isomorphic to the alternating group A_5 (cf. [**34**, Ch. 1]). It therefore can be presented as $\mathcal{I} = \langle g_2, g_3 \,|\, g_2^2, g_3^3, (g_3g_2)^5\rangle$.

Other examples of finite rotation symmetry groups are the cyclic group C_n, $n \geqslant 1$, and the dihedral group D_m of order $2m$, $m \geqslant 1$. C_n is the rotation symmetry group of the n-gonal regular pyramid, which has a regular n-gon as base and where the axis through the apex and the centre of the base is perpendicular to the base, see Figure 2.5. It is generated by the rotation about this axis by an angle of $2\pi/n$.

The dihedral group D_m is the rotation symmetry group of a right prism with a regular m-gon as base (or alternatively, of an m-gonal dipyramid), see Figure 2.6. It is generated by two elements, namely the rotation by $2\pi/m$ about the axis joining the centres of the two m-gonal faces and by the rotation by π about an axis that is an axis of reflection of the two-dimensional m-gon lying parallel to and at equal distance from the two m-gonal faces of the prism. The latter rotation interchanges the two m-gonal faces. Note that D_n is also the full symmetry group (including orientation-reversing isometries) of the planar regular m-gon which is generated by the rotation about the centre by $2\pi/m$ and the reflection in a suitable line through the centre.

REMARK 2.22. Any isometry $f\colon \mathbb{R}^3 \to \mathbb{R}^3$ with $f(0) = 0$ is a linear isomorphism (cf. [**43**, Ch. 6.3]). As an isometry, f preserves distances, i.e. $\|f(x) - f(y)\| = \|x - y\|$ for all $x, y \in \mathbb{R}^3$. One easily receives $\langle f(x), f(y)\rangle = \langle x, y\rangle$, where $\langle\,\cdot\,\rangle$ denotes the standard scalar product on $\mathbb{R}^3$. Thus f is orthogonal. Given a Platonic solid (or a regular pyramid,

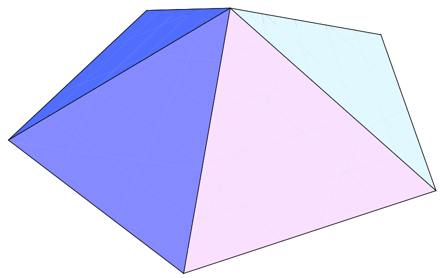

FIGURE 2.5. 5-gonal regular pyramid

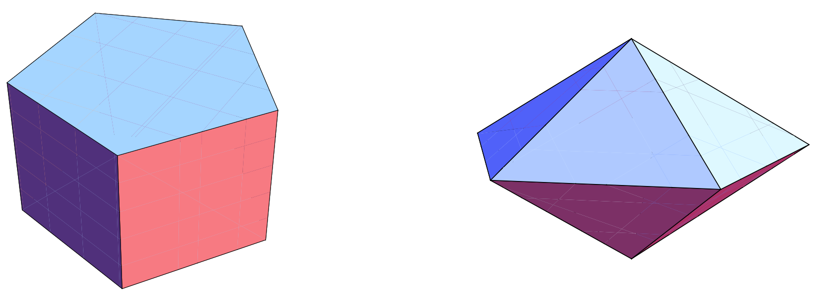

FIGURE 2.6. 5-gonal regular prism and 5-gonal regular dipyramid

or a right prism) in $\mathbb{R}^3$ that is centred at the origin, then every element of its rotation symmetry group leaves the origin fixed and is therefore an orthogonal linear transformation. The representing matrix with respect to the canonical basis of $\mathbb{R}^3$ is thus an element of $\mathrm{SO}(3,\mathbb{R})$.

As the next theorem indicates, the examples of finite rotation symmetry groups given above are the only finite groups of rotations of 3-space.

THEOREM 2.23. [**34**, Thm. 11] *Every finite subgroup of* $\mathrm{SO}(3,\mathbb{R})$ *is isomorphic to exactly one of the following groups.*

$$C_n,$$

$$\begin{gathered} D_m, \\ \mathcal{T} \simeq A_4, \\ \mathcal{O} \simeq S_4, \\ \mathcal{I} \simeq A_5, \end{gathered}$$

where $n \geqslant 1$ and $m \geqslant 2$. □

Thus the classification of the finite subgroups of $\mathrm{SO}(3, \mathbb{Q})$ reduces to the question which of the above groups are contained in $\mathrm{SO}(3, \mathbb{Q})$. As seen above, the cyclic group C_n is the rotation symmetry group of a regular pyramid in 3-space centred at the origin with the base consisting of a regular n-gon situated parallel to the x-y-plane. A generator is the rotation d_n about the z-axis by $2\pi/n$, in matrix form

$$d_n = \begin{pmatrix} \cos(\frac{2\pi}{n}) & -\sin(\frac{2\pi}{n}) & 0 \\ \sin(\frac{2\pi}{n}) & \cos(\frac{2\pi}{n}) & 0 \\ 0 & 0 & 1 \end{pmatrix} \in \mathrm{SO}(3, \mathbb{R}).$$

Clearly, the identity d_1 is contained in $\mathrm{SO}(3, \mathbb{Q})$ as are d_2 and d_4, because one has

$$d_2 = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad d_4 = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

For which other values of n is d_n an element of $\mathrm{SO}(3, \mathbb{Q})$? A necessary condition is that the trace of d_n lies in $\mathbb{Q}$, or equivalently, $2\cos(2\pi/n) \in \mathbb{Q}$.

The field $\mathbb{Q}(2\cos(2\pi/n)) = \mathbb{Q}(\zeta_n + \overline{\zeta_n})$ is the maximal real subfield of the n-th cyclotomic field $\mathbb{Q}(\zeta_n)$. For $n \geqslant 3$ this implies $\varphi(n) = 2$ by Corollary 2.3 and thus $n \in \{3, 4, 6\}$. For $n = 3$ and $n = 6$, $d_n \notin \mathrm{SO}(3, \mathbb{Q})$, because in each case entries of the matrix involve a rational multiple of $\sqrt{3}$. But rotating the pyramid around the origin such that the axis ℓ through the apex and the centre of the n-gon goes through $(1, 1, 1)^t$, the rotation symmetry group C_n is generated by the rotation by $2\pi/n$ about ℓ, in matrix form

$$d_3' = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \quad \text{and} \quad d_6' = \frac{1}{3}\begin{pmatrix} 2 & -1 & 2 \\ 2 & 2 & -1 \\ -1 & 2 & 2 \end{pmatrix} \in \mathrm{SO}(3, \mathbb{Q}).$$

Therefore C_1, C_2, C_3, C_4 and C_6 are the only finite cyclic subgroups of $\mathrm{SO}(3, \mathbb{Q})$.

Similarly, we proceed for the rotation symmetry group D_m of a right m-sided prism centred at the origin in 3-space whose two m-gonal faces lie suitably parallel to the x-y-plane. Then D_m is generated by d_m and the rotation by π about the x-axis, in matrix form

$$j = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

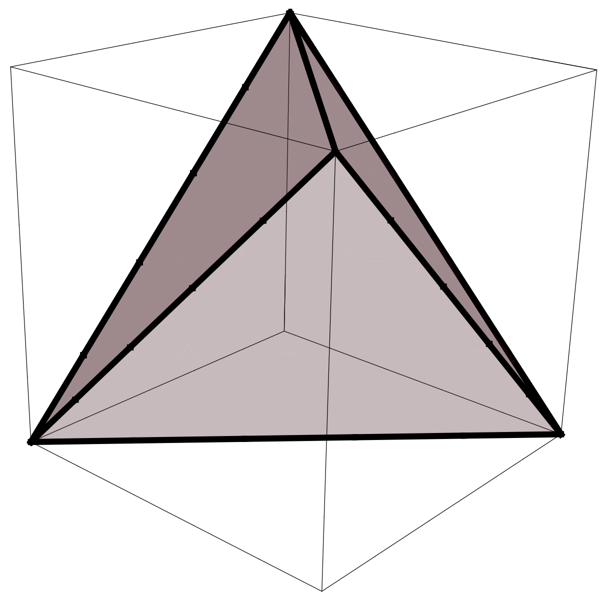

FIGURE 2.7. A tetrahedron inscribed in a cube

For $m = 3$ and $m = 6$, we consider again a rotated version of the prism such that its rotation symmetry group is generated by d'_m and by

$$j' = \frac{1}{3}\begin{pmatrix} -2 & 1 & -2 \\ 1 & -2 & -2 \\ -2 & -2 & 1 \end{pmatrix} \in \mathrm{SO}(3, \mathbb{Q}).$$

Hence the only dihedral groups D_m that are subgroups of $\mathrm{SO}(3, \mathbb{Q})$ are $D_1 \simeq C_2$, D_2, D_3, D_4 and D_6.

The cube of edge length 2 with Cartesian coordinates $(\pm 1, \pm 1, \pm 1)^t$ is centred at the origin and its rotation symmetry group $\mathcal{O}$ is generated by the following two rotations: the rotation r by $2\pi/3$ about the (body) diagonal joining the vertices $(-1, -1, -1)^t$ and $(1, 1, 1)^t$, and the rotation d_4 about the z-axis by $\pi/2$. In matrix form these are

$$r = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} (= d'_3) \quad \text{and} \quad d_4 = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

This shows that $\mathcal{O}$ is a subgroup of $\mathrm{SO}(3, \mathbb{Q})$. Moreover, one even has $\mathcal{O} \simeq \mathrm{SO}(3, \mathbb{Z})$ (cf. [**3**]).

One can embed a tetrahedron into the cube above by taking its vertices to be $(1, 1, 1)^t$, $(1, -1, -1)^t$, $(-1, 1, -1)^t$ and $(-1, -1, 1)^t$. Each vertex is also a vertex of the cube and each edge is the diagonal of a face of the cube, see Figure 2.7. The rotation symmetry group $\mathcal{T}$ of this tetrahedron is generated by the rotation r from above, whose rotation axis goes through the vertex $(1, 1, 1)^t$ and the opposite face centre of the tetrahedron, and the rotation d_2 about the z-axis that pierces the midpoints of two opposite edges. Thus $\mathcal{T}$ is a subgroup of $\mathrm{SO}(3,\mathbb{Q})$ and moreover, $\mathcal{T} \subset \mathcal{O}$.

This leaves only the icosahedral group $\mathcal{I}$ to consider. It contains a rotation of order 5. Since all finite elements of $\mathrm{SO}(3,\mathbb{Q})$ have order $1,2,3,4$ or 6 (cf. Lemma 2.36), $\mathcal{I}$ is not a subgroup of $\mathrm{SO}(3,\mathbb{Q})$. As mentioned before, the distances between some vertices of the icosahedron involve the golden ratio τ. So it is not surprising that $\mathcal{I}$ is actually a subgroup of $\mathrm{SO}(3,\mathbb{Q}(\tau))$. For instance, consider the icosahedron given by the 12 vertices

$$\begin{aligned}&(0,\pm1,\pm\tau)^t,\\&(\pm1,\pm\tau,0)^t,\\&(\pm\tau,0,\pm1)^t,\end{aligned}$$

which is depicted in Figure 2.4. It has edge length 2 and is centred at the origin. The y-axis goes through midpoints of two opposite edges which makes it a rotation axis of order 2. The axis through the origin and $(0,\tau,-\tau')^t$, where $\tau' = 1-\tau$ is the algebraic conjugate of τ, pierces the icosahedron at the centres of two opposite faces. Therefore, the rotation v about this axis by $2\pi/3$ is an element of the symmetry group. Denote by w the rotation by π about the y-axis, then the two rotations in matrix form are

$$w = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \quad \text{and} \quad v = \frac{1}{2}\begin{pmatrix} -1 & \tau' & \tau \\ -\tau' & \tau & 1 \\ -\tau & 1 & \tau' \end{pmatrix}.$$

Both are elements of $\mathrm{SO}(3,\mathbb{Q}(\tau))$. By also taking into account the group presentation $\mathcal{I} = \langle g_2, g_3 \,|\, g_2^2, g_3^3, (g_3 g_2)^5 \rangle$, one verifies that v and w generate $\mathcal{I}$. Note that vw is the rotation by $2\pi/5$ about the axis through the origin and the vertex $(1,-\tau,0)^t$.

To conclude, we summarise as follows.

Corollary 2.24. *The only finite subgroups of* $\mathrm{SO}(3,\mathbb{Q})$ *are the trivial group,* C_2*,* C_3*,* C_4*,* C_6*,* D_2*,* D_3*,* D_4*,* D_6*,* $\mathcal{T} \simeq A_4$ *and* $\mathcal{O} \simeq S_4$ *(where we omitted* D_1 *due to* $C_2 \simeq D_1$*).*

2.4. Generalised dihedral groups

Since we shall make use of the results of [**37**] repeatedly, we summarise them in some detail. In their article, Radin and Sadun classify all subgroups of $\mathrm{SO}(3,\mathbb{R})$ that are generated by two rotations of finite order such that the rotation axes of these generators enclose an angle of finite order. They call such groups *generalised dihedral groups.* Mostly, these groups are free products, or amalgamated free products of cyclic or dihedral groups; cf. Section 2.1.3.

To be more concrete, we adopt the notation of the article and denote by ℓ the line in the x-y-plane of $\mathbb{R}^3$ through the origin that makes an angle of $2\pi n/m$ with the x-axis (where $n,m \in \mathbb{N}$ with $\gcd(n,m)=1$ and $m>2$). For $p \in \mathbb{N}$ let $R_x^{2\pi/p}$ be the rotation by $2\pi/p$ about the x-axis and define $R_\ell^{2\pi/q}$ for $q \in \mathbb{N}$ accordingly, see Figure 2.8 for an illustration.

Define $G_{n/m}(p,q)$ to be the subgroup of $\mathrm{SO}(3,\mathbb{R})$ generated by $R_x^{2\pi/p}$ and $R_\ell^{2\pi/q}$. From three simple facts on rotations by multiples of $\pi/2$, the authors deduce all relations between

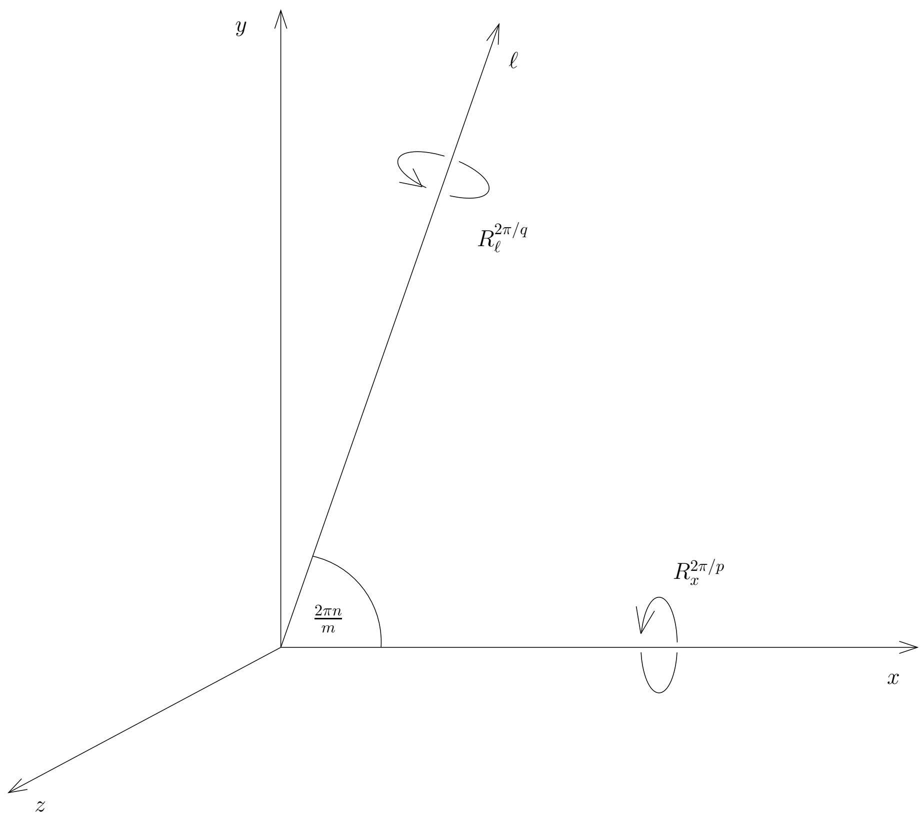

FIGURE 2.8. The generators of $G_{n/m}(p, q)$.

the two generators $R_x^{2\pi/p}$ and $R_\ell^{2\pi/q}$, and thereby obtain a group presentation for $G_{n/m}(p, q)$. These simple facts are the following.

$$R_x^\pi R_y^\theta R_x^\pi = R_y^{-\theta}, \qquad \text{where } 0 \leqslant \theta < 2\pi \tag{2.6}$$

$$R_y^{\pi/2} R_x^{\pi/2} R_y^{\pi/2} = R_x^{\pi/2} R_y^{\pi/2} R_x^{\pi/2} \tag{2.7}$$

$$R_\ell^\pi R_x^\pi = R_z^{4\pi n/m} \tag{2.8}$$

The first result concerns those generalised dihedral groups where the rotation axes of the two generators are separated by an angle of $\pi/2$.

THEOREM 2.25. [**37**, Thm. 2] *The group $G_{1/4}(p, q)$ has the following structure:*

(1) *If p and q are both odd, then*

$$G_{1/4}(p, q) \simeq C_p \star C_q.$$

(2) *If p is even and q is odd, then*

$$G_{1/4}(p, q) \simeq C_p \star_{C_2} D_q.$$

(3) *If p and q are both even, but q is not divisible by 4, then*

$$G_{1/4}(p,q) \simeq D_p \star_{D_2} D_q.$$

(4) *If p and q are both divisible by 4, then*

$$G_{1/4}(p,q) \simeq D_{\mathrm{lcm}(p,q)} \star_{D_4} S_4.$$

□

Note that the rotation symmetry groups of most of the polyhedra from Section 2.3 arise as generalised dihedral groups, namely S_4 as well as the cyclic groups C_p and the dihedral groups D_p, $p \in \mathbb{N}$. One has

$$G_{1/4}(p,1) \simeq C_p \star C_1 \simeq C_p$$

when p is odd. Otherwise, Lemma 2.18 implies

$$G_{1/4}(p,1) \simeq C_p \star_{C_2} D_1 \simeq C_p \star_{C_2} C_2 \simeq C_p.$$

The dihedral group D_p of order $2p$ is realized for $q = 2$: If p is odd, Lemma 2.18 yields

$$G_{1/4}(p,2) \simeq C_2 \star_{C_2} D_p \simeq D_p,$$

and if p is even, the same Lemma shows

$$G_{1/4}(p,2) \simeq D_p \star_{D_2} D_2 \simeq D_p = \langle \alpha, \beta \mid \alpha^p, \beta^2, (\alpha\beta)^2 \rangle.$$

The relation $(\alpha\beta)^2$ in the presentation of D_p is equivalent to (2.6).

Furthermore, one has $G_{1/4}(4,4) \simeq D_4 \star_{D_4} S_4 \simeq S_4$. By Lemma 2.10, S_4 has the presentation $S_4 = \langle u, v \mid u^4, v^4, (u^2v)^2, (uv^2)^2, (uv)^3 \rangle$. Here $(u^2v)^2$ and $(uv^2)^2$, respectively, are consequences of (2.6), whereas the last relation $(uv)^3$ is a consequence of (2.7). But not all rotation symmetry groups of Platonic solids appear.

Lemma 2.26. *The rotation symmetry group of the tetrahedron A_4 and the rotation symmetry group of the icosahedron A_5 can each be generated by two rotations of finite order, but only with axes separated by an angle of infinite order. Thus, neither A_4 nor A_5 is a generalised dihedral group.*

Proof. We show that any two rotations of A_4 whose rotation axes are separated by an angle of finite order do not generate A_4. Consider the angles between the rotation axes of elements of the tetrahedral group A_4. The only angles of finite order that appear are the ones between two axes of 2-fold symmetry passing from the midpoint of an edge of the tetrahedron to the midpoint of the opposite edge. Two such axes form an orthogonal angle. The corresponding two rotations by π generate a group isomorphic to $G_{1/4}(2,2) \simeq D_2$. Thus A_4 is not a generalised dihedral group.

The elements of the rotation symmetry group A_5 of the icosahedron have been described in Section 2.3. One finds that there are neither angles of finite order between two axes of 5-fold symmetry, nor between two axes of 3-fold symmetry, nor between an axis of 5-fold and one of 3-fold symmetry. The only angles of finite order between two axes of 2-fold symmetry are orthogonal ones, or integer multiples of $\pi/5$, or integer multiples of $\pi/3$. In the first case, the subgroup of A_5 generated by the two rotations of order 2 is the dihedral group $D_2 \subsetneq A_5$. In the second case, one checks that the subgroup generated by the two corresponding rotations of order 2 does not contain any elements of order 3. Therefore it does not coincide with A_5. In the last case, one deduces that the corresponding 2-generator subgroup of A_5 does not contain any elements of order 5 and hence is not A_5.

The only angle of finite order between an axis of 5-fold and one of 2-fold symmetry is orthogonal. Corresponding rotations generate $G_{1/4}(5,2) \simeq D_5 \subsetneq A_5$. The last case is the one with one axis of 3-fold and one axis of 2-fold symmetry. Here, the only angle of finite order between such axes is orthogonal as well. Hence, corresponding rotations generate $G_{1/4}(3,2) \simeq D_3 \subsetneq A_5$. □

EXAMPLE 2.27. As seen in Section 2.3, the icosahedral group A_5 is a subgroup of $\mathrm{SO}(3,\mathbb{R})$ and it is generated by the following pair of rotations,

$$w = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \quad \text{and} \quad v = \frac{1}{2}\begin{pmatrix} -1 & \tau' & \tau \\ -\tau' & \tau & 1 \\ -\tau & 1 & \tau' \end{pmatrix}.$$

With the notation of this section, one has $w = R_y^{\pi}$ and $v = R_k^{2\pi/3}$, where k is the axis through the origin and $(0,\tau,-\tau')^t$. The y-axis and the axis k are not, of course, separated by an angle of finite order (cf. Lemma 2.26). Indeed, the angle ψ between these two axes is $\arccos(\tau/\sqrt{3})$. Can we chose two generators of A_5 such that the angle between the two rotation axes is "simpler" in some way? Consider the two elements $g := vw$ and $h := vwv$ of the icosahedral group. Then g is the rotation by $2\pi/5$ about the axis trough the origin and $(1,-\tau,0)^t$, whereas h is the rotation by $2\pi/5$ about the axis through the origin and $(0,1,\tau)^t$. Due to $v = g^{-1}h$ and $w = (g^{-1}h)^2 g$, g and h generate the icosahedral group. The angle ϕ between their rotation axes satisfies $\cos(\phi) = -1/\sqrt{5}$, and hence, in contrast to ψ, one has $(\cos(\phi))^2 \in \mathbb{Q}$. This implies that $e^{2i\phi}$ is a quadratic irrational, i.e. $e^{2i\phi}$ is an irrational number and a root of a quadratic polynomial with integer coefficients. Angles of this type are called *geodetic* in **[16]**. For generalised dihedral groups, there are two generators with rotation axes separated by an angle of finite order, i.e. an angle α which is a rational multiple of π, or equivalently, an angle α for which $e^{2i\alpha}$ is a root of unity. Thus one can view geodetic angles as a generalisation of angles of finite order. Summing up, the icosahedral group A_5 is not a generalised dihedral group, but it can be generated by a pair of finite order rotations such that the rotation axes of these generators form a geodetic angle.

THEOREM 2.28. [**37**, Thm. 3] *If p and q are odd, or if p is even, q is odd and $m \neq 4$, then one has*

$$G_{n/m}(p,q) \simeq C_p \star C_q.$$

If p is even, q is odd and $m = 4$, then

$$G_{n/m}(p, q) \simeq C_p \star_{C_2} D_q.$$

□

The cases that remain are those where p and q are both even. For any integer k let $\varrho(k)$ be the number of powers of 2 that divide k, e.g. $\varrho(6) = 1$ and $\varrho(9) = 0$.

THEOREM 2.29. [**37**, Thm. 4] *If*

(1) *p is even, q is even and m is odd, or*
(2) *$\varrho(p) \geqslant 2$, $\varrho(q) \geqslant 2$ and $\varrho(m) = 2$, or*
(3) *q is even and $\varrho(p) \geqslant \varrho(m) \geqslant 3$,*

then one has

$$G_{n/m}(p, q) \simeq G_{1/4}\big(\operatorname{lcm}(p, q), m\big).$$

□

Now there are only some cases left where p and q are even and m is divisible by 4. Without loss of generality, assume $\varrho(p) \geqslant \varrho(q)$.

THEOREM 2.30. [**37**, Thm. 5]

(1) *If $\varrho(m) > \varrho(p) = \varrho(q) = 1$, then*
$$G_{n/m}(p, q) = \langle \alpha, \beta, \gamma \mid \alpha^p, \gamma^q, \beta^2, (\alpha\beta)^2, (\gamma\beta)^2, (\alpha^{p/2}\gamma^{q/2})^{m/4}\beta \rangle.$$
(2) *If $\varrho(m) > \varrho(p) > 1$ and $\varrho(m) > \varrho(q) > 1$, then*
$$G_{n/m}(p, q) \simeq G_{1/4}(r, 4) \star_{D_r} G_{1/4}(r, 4),$$
where $r = \operatorname{lcm}(p, q, m/2)$.
(3) *If $\varrho(m) > \varrho(p) > \varrho(q) = 1$, then*
$$G_{n/m}(p, q) \simeq G_{1/4}\big(\operatorname{lcm}(p, m/2), 4\big) \star_{D_2} D_q.$$
(4) *If $\varrho(p) > \varrho(q) = 1$ and $\varrho(m) = 2$, then*
$$G_{n/m}(p, q) \simeq D_p \star_{D_2} D_{\operatorname{lcm}(q, m/2)}.$$

□

Note that $G_{n/m}(p, q)$ as in Theorem 2.30(1) is a quotient of the amalgamated free product $D_p \star_{C_2} D_q$ by the last relation, cf. Theorem 2.15.

In most cases, $G_{n/m}(p, q)$ is infinite (cf. Section 2.1.3), the only exceptions being the rotation symmetry groups of the cube, the p-gonal regular pyramid and the p-gonal regular prism, where $p \in \mathbb{N}$. Together with Lemma 2.26 and the classification of finite subgroups of SO(3, $\mathbb{R}$) from Theorem 2.23, this implies the following.

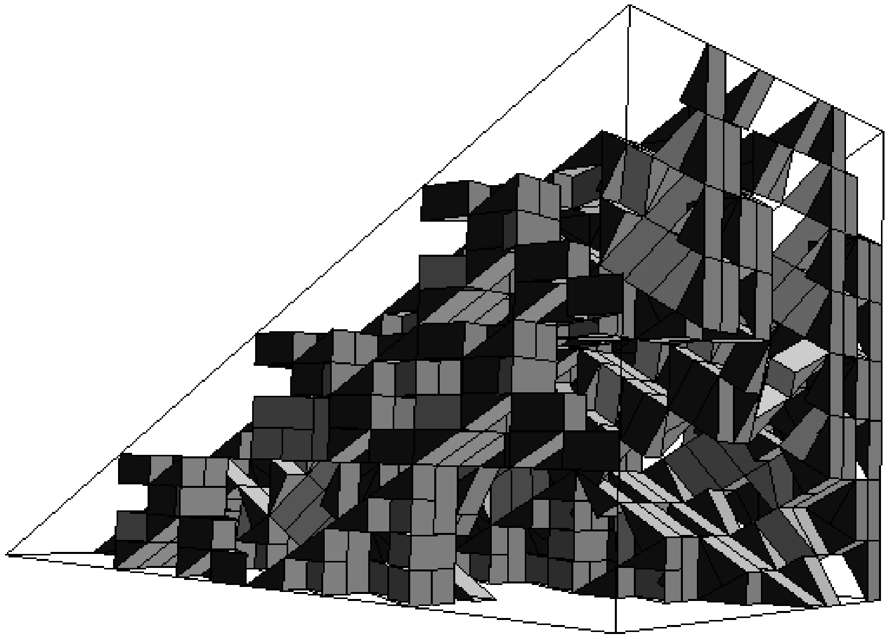

FIGURE 2.9. Part of a quaquaversal tiling (picture taken from [**38**])

LEMMA 2.31. *The only finite subgroups of* $\mathrm{SO}(3,\mathbb{R})$ *are the alternating groups* A_4 *and* A_5 *and the finite generalised dihedral groups (i.e. those finite subgroups of* $\mathrm{SO}(3,\mathbb{R})$ *that are generated by two rotations of finite order such that the rotation axes of these generators also enclose an angle of finite order).* □

REMARK 2.32. The generalised dihedral groups play a role in the theory of tilings of Euclidean 3-space. An example called "quaquaversal tilings" was constructed in [**15**]. A quaquaversal tiling of $\mathbb{R}^3$ consists of congruent copies of a single triangular prism that appears in an infinite number of orientations. This set of orientations forms a dense subgroup of $\mathrm{SO}(3,\mathbb{R})$ that, in fact, is the generalised dihedral group $G_{1/4}(6,4)$ [**39**]. The quaquaversal tiling can be viewed as a 3-dimensional extension of the pinwheel tiling of the plane. Another example is a 3-dimensional "dite and kart tiling" constructed in [**39**], which is based on a version of the 2-dimensional kite and dart tiling and has $G_{1/4}(10,4)$ as orientation group.

2.5. Cayley's parametrisation

We recall some results on quaternions which can be found in [**26, 29, 10**]. Let K be a real field. The corresponding Hamiltonian quaternion algebra is

$$\mathbb{H}(K) = Ke + Ki + Kj + Kk \tag{2.9}$$

where the defining relations for the generating elements e, i, j, k are given by

$$i^2 = j^2 = k^2 = -e \quad \text{and} \quad ij = -ji = k. \tag{2.10}$$

$\mathbb{H}(K)$ is a skew field with unit element e and its elements are called *quaternions*. For a quaternion q it is customary to write $q = (x_0, x_1, x_2, x_3)$ instead of $q = x_0e + x_1i + x_2j + x_3k$. Assign to a nonzero quaternion $\rho = (\kappa, \lambda, \mu, \nu)$ the following matrix $R(\rho)$ in $\mathrm{SO}(3, K)$:

$$R(\rho) = \frac{1}{|\rho|^2}\begin{pmatrix} \kappa^2+\lambda^2-\mu^2-\nu^2 & -2\kappa\nu+2\lambda\mu & 2\kappa\mu+2\lambda\nu \\ 2\kappa\nu+2\lambda\mu & \kappa^2-\lambda^2+\mu^2-\nu^2 & -2\kappa\lambda+2\mu\nu \\ -2\kappa\mu+2\lambda\nu & 2\kappa\lambda+2\mu\nu & \kappa^2-\lambda^2-\mu^2+\nu^2 \end{pmatrix}, \tag{2.11}$$

where $|\rho|^2 = \kappa^2 + \lambda^2 + \mu^2 + \nu^2$. For $K = \mathbb{R}$, the next result goes back to Euler in 1770.

THEOREM 2.33. *Let K be a real algebraic number field. Then every rotation matrix $M \in \mathrm{SO}(3, K)$ is of the form $M = R(\rho)$ for some nonzero $\rho \in \mathbb{H}(K)$, where $R(\rho)$ is defined as in* (2.11). *The mapping $\rho \mapsto R(\rho)$ is called* Cayley's parametrisation of $\mathrm{SO}(3, K)$.

PROOF. Consider the map

$$\begin{aligned} \mathbb{H}(\mathbb{R})^\bullet &\longrightarrow \mathrm{SO}(3, \mathbb{R}), \\ q &\longmapsto R(q). \end{aligned}$$

It is a group epimorphism whose kernel is $\mathbb{R}e \setminus \{0\}$; cf. [**26**, Ch. 3.6]. Moreover, one has $R(t \cdot q) = R(q)$ for all $t \in \mathbb{R}^\bullet$. Let $M \in \mathrm{SO}(3, K)$. Then there is a nonzero quaternion $\rho = (\kappa, \lambda, \mu, \nu) \in \mathbb{H}(\mathbb{R})$ such that $R(\rho) = M$.

If $\kappa, \lambda, \mu, \nu \in K$, there is nothing to prove. If, without loss of generality, $\kappa \in \mathbb{R} \setminus K$, consider the matrix entries of $R(\rho)$. They all lie in K, hence one also has

$$\frac{\lambda\nu}{|\rho|^2}, \frac{\mu\nu}{|\rho|^2}, \frac{\kappa\nu}{|\rho|^2}, \frac{\kappa\lambda}{|\rho|^2} \in K.$$

Thus $\lambda/\kappa, \mu/\kappa, \nu/\kappa \in K$ which implies $\lambda, \mu, \nu \in \kappa K$. Since there exists $\ell \in \mathbb{R}^\bullet$ with $\ell\kappa \in K^\bullet$, one has $\ell \cdot \rho = (\ell\kappa, \ell\lambda, \ell\mu, \ell\nu) \in \mathbb{H}(K)^\bullet$ with $R(\ell \cdot \rho) = R(\rho) = M$. $\square$

Thus, in order to obtain all rotation matrices of $\mathrm{SO}(3, K)$, where K is a real algebraic number field, it suffices to consider all nonzero quaternions $\rho = (\kappa, \lambda, \mu, \nu)$ such that $\kappa, \lambda, \mu, \nu \in K$ and apply the mapping given by (2.11). But like this, any matrix in $\mathrm{SO}(3, K)$ is obtained multiple times since $R(t \cdot \rho) = R(\rho)$ for all nonzero $t \in K$. This immediately suggests the question whether it is possible to further restrict the entries of ρ and still obtain all elements of $\mathrm{SO}(3, K)$.

For the remainder of the section, let K be a real algebraic number field of class number one, meaning that the ring of integers $\mathcal{O}_K$ of K is a principal ideal domain. A *greatest common divisor of* $a, b \in \mathcal{O}_K$ is an element $d \in \mathcal{O}_K$ that is a divisor of both a and b such that every common divisor of a and b is also a divisor of d. (Since $\mathcal{O}_K$ is a principal ideal domain, the greatest common divisor of a and b can also be characterised as a generator of the ideal that is generated by a and b.) Any two elements $a, b \in \mathcal{O}_K$, not both zero, have a greatest common divisor d and it is unique only up to associates (cf. [**28**, Ch. 1.5]). We denote it by $\gcd(a,b)$. Similarly, any two elements $a,b \in \mathcal{O}_K$ have a least common multiple in $\mathcal{O}_K$, denoted by $\mathrm{lcm}(a,b)$.

Definition 2.34. A quaternion $\rho = (\kappa,\lambda,\mu,\nu) \in \mathbb{H}(K)$ is called *$\mathcal{O}_K$-primitive*, if $\kappa,\lambda,\mu,\nu \in \mathcal{O}_K$ and $\gcd(\kappa,\lambda,\mu,\nu)$ is a unit in $\mathcal{O}_K$.

Theorem 2.35. *Let K be a real algebraic number field of class number one. Then* $\mathrm{SO}(3,K)$ *is parametrised by the set of $\mathcal{O}_K$-primitive quaternions. Moreover, $R(\rho) = R(\rho')$ holds for two $\mathcal{O}_K$-primitive quaternions if and only if $\rho' = \varepsilon \cdot \rho$ for some unit ε of $\mathcal{O}_K$.*

Proof. Let $M \in \mathrm{SO}(3,K)$. By Theorem 2.33, there exists $q \in \mathbb{H}(K)$ with $R(\rho) = M$. Since K is the field of fractions of $\mathcal{O}_K$, one has $\kappa = \kappa_1/\kappa_2$, $\lambda = \lambda_1/\lambda_2$, $\mu = \mu_1/\mu_2$, and $\nu = \nu_1/\nu_2$ for some $\kappa_i,\lambda_i,\mu_i,\nu_i \in \mathcal{O}_K$, $i \in \{1,2\}$. For $\ell := \mathrm{lcm}(k_2,\lambda_2,\mu_2,\nu_2)$, the entries of the quaternion $\ell \cdot q$ are all in $\mathcal{O}_K$ and $R(\ell \cdot q) = R(q)$. Setting $g = \gcd(\ell\kappa,\ell\lambda,\ell\mu,\ell\nu)$, the quaternion $(\ell/g)\cdot q$ is $\mathcal{O}_K$-primitive and satisfies $R((\ell/g)\cdot q) = R(q) = M$.

It remains to show that two $\mathcal{O}_K$-primitive quaternions ρ,ρ' parametrise the same matrix in $\mathrm{SO}(3,K)$ if and only if $\rho' = \varepsilon\cdot\rho$ for some unit ε of $\mathcal{O}_K$. If the latter holds, then we already know $R(\rho') = R(\varepsilon\cdot\rho) = R(\rho)$. Conversely, let $\rho' = (\kappa',\lambda',\mu',\nu')$ and $\rho = (\kappa,\lambda,\mu,\nu)$ be two $\mathcal{O}_K$-primitive quaternions such that $R(\rho) = R(\rho')$. The mapping $\mathbb{H}(\mathbb{R})^\bullet \longrightarrow \mathrm{SO}(3,\mathbb{R})$, $q \mapsto R(q)$ is a group epimorphism on the multiplicative group of $\mathbb{H}(\mathbb{R})$ whose kernel is $\mathbb{R}e \setminus \{0\}$; cf. [**26**, Ch. 3.6]. Thus $R(\rho) = R(\rho')$ implies $\rho'\rho^{-1} \in \mathbb{R}e \setminus \{0\}$, say $\rho'\rho^{-1} = t\cdot e$, $t \in \mathbb{R}^\bullet$. Hence

$$\rho' = t\cdot e\cdot\rho = t\cdot\rho \tag{2.12}$$

yields $t\kappa, t\lambda, t\mu, t\nu \in \mathcal{O}_K$. Therefore $t \in K^\bullet$. Let $t_1,t_2 \in \mathcal{O}_K$ with $t = t_1/t_2$. By (2.12) one has $(t_1/t_2)\kappa = \kappa'$, implying that t_2 divides $t_1\kappa$ in $\mathcal{O}_K$. Similarly, t_2 also divides $t_1\lambda, t_1\mu$ and $t_1\nu$. Thus t_2 divides $\gcd(t_1\kappa,t_1\lambda,t_1\mu,t_1\nu)$. By straightforward calculations one finds

$$\gcd(t_1\kappa,t_1\lambda,t_1\mu,t_1\nu) = t_1\gcd(\kappa,\lambda,\mu,\nu) = t_1\varepsilon$$

for some unit ε of $\mathcal{O}_K$, because ρ is $\mathcal{O}_K$-primitive. Hence $t \in \mathcal{O}_K$. From (2.12) one deduces that t divides $\gcd(\kappa',\lambda',\mu',\nu')$ which is a unit in $\mathcal{O}_K$ since ρ' is $\mathcal{O}_K$-primitive. This shows that t itself is a unit of $\mathcal{O}_K$ with $\rho' = t\cdot\rho$. □

The identity matrix E_3 is only parametrised by the nonzero quaternions of the form $(\kappa,0,0,0) \in \mathbb{H}(K)$. A useful property of Cayley's parametrisation is that for any other nonzero quaternion $\rho = (\kappa,\lambda,\mu,\nu)$ the vector $(\lambda,\mu,\nu)^t$ gives the rotation axis of $R(\rho)$. Moreover, the rotation angle ϕ of $R(\rho)$ is given by

$$\cos(\phi) = \frac{\kappa^2-\lambda^2-\mu^2-\nu^2}{\kappa^2+\lambda^2+\mu^2+\nu^2}, \tag{2.13}$$

due to $\mathrm{tr}(R(\rho)) = 1 + 2\cos(\phi)$.

2.5.1. Cayley's parametrisation of $\mathrm{SO}(3,\mathbb{Q})$**.** Via Cayley parametrisation with $\mathbb{Z}$-primitive quaternions, every matrix of $\mathrm{SO}(3,\mathbb{Q})$ is obtained exactly twice, namely as $R(\rho) = R(-\rho)$, cf. Theorem 2.35. For shortness, we call $\mathbb{Z}$-primitive quaternions *primitive.*

LEMMA 2.36. *Let R be an element of* $\mathrm{SO}(3,\mathbb{Q})$ *of finite order. Then R is a rotation of order* $1, 2, 3, 4$ *or* 6.

PROOF. We use Cayley's parametrisation of $\mathrm{SO}(3,\mathbb{Q})$ with primitive quaternions. Let $\rho = (\kappa, \lambda, \mu, \nu) \in \mathbb{H}(\mathbb{Q})$ be a primitive quaternion such that $R(\rho) = R$. Since R has finite order, its rotation angle $\phi \in [0, 2\pi)$ can be expressed as $\phi = 2\pi n/m$, where $\gcd(n, m) = 1$. Equation (2.13) implies

$$\cos(\phi) = \frac{\kappa^2 - \lambda^2 - \mu^2 - \nu^2}{\kappa^2 + \lambda^2 + \mu^2 + \nu^2} \quad \in \mathbb{Q}.$$

One has $\mathbb{Q} = \mathbb{Q}(\cos(2\pi n/m)) = \mathbb{Q}(2\cos(2\pi/m))$, the maximal real subfield of $\mathbb{Q}(\zeta_m)$. Therefore one has $\varphi(m)/2 = 1$ by Corollary 2.3, if $m \geqslant 3$. $\square$

Thus, the only rotations of interest to the problem at hand are those of order $1, 2, 3, 4$ and 6. Consider the following equations that we shall use repeatedly.

$$\rho = (0, \lambda, \mu, \nu) \tag{0}$$

$$\kappa^2 = \lambda^2 + \mu^2 + \nu^2 \tag{I}$$

$$\kappa^2 = 3(\lambda^2 + \mu^2 + \nu^2) \tag{II}$$

$$3\kappa^2 = \lambda^2 + \mu^2 + \nu^2 \tag{III}$$

LEMMA 2.37. *Equation* (II) *has an integer solution if and only if Equation* (III) *has an integer solution.*

PROOF. If $\kappa, \lambda, \mu, \nu$ is an integer solution of Equation (II), then it is easy to verify that $\kappa, 3\lambda, 3\mu, 3\nu$ is an integer solution of Equation (III). If conversely $\kappa', \lambda', \mu', \nu'$ is an integer solution of Equation (III), then $3\kappa', \lambda', \mu', \nu'$ is one of Equation (II). $\square$

LEMMA 2.38. *If $\rho = (\kappa, \lambda, \mu, \nu)$ is a primitive quaternion, then the following statements hold.*

(1) *$R(\rho)$ is a rotation of order 1 if and only if $\rho = (\pm 1, 0, 0, 0)$.*
(2) *$R(\rho)$ is a rotation of order 2 if and only if $\kappa = 0$.*
(3) *$R(\rho)$ is a rotation of order 4 if and only if ρ satisfies Equation* (I).
(4) *$R(\rho)$ is a rotation of order 6 if and only if ρ satisfies Equation* (II).
(5) *$R(\rho)$ is a rotation of order 3 if and only if ρ satisfies Equation* (III).

PROOF. (1) If $R(\rho)$ has order 1, then its rotation angle ϕ satisfies $\cos(\phi) = 1$ and Equation (2.13) yields $\rho = (\pm 1, 0, 0, 0)$, because ρ is primitive. On the other hand, one easily verifies $R((\pm 1, 0, 0, 0)) = E_3$ by (2.11), where E_3 denotes the identity matrix.

(2) If $R(\rho)$ has order 2, then its rotation angle is π. We use $\cos(\pi) = -1$ in Equation (2.13) to obtain $\kappa = 0$. Conversely, $\kappa = 0$ implies $\cos(\varphi) = -1$ for the rotation angle $\varphi \in [0, 2\pi)$ of $R(\rho)$. Hence $\varphi = \pi$, so $R(\rho)$ has order 2.

(3) If $R(\rho)$ has order 4, then straightforward calculations for its rotation angle ϕ show that $\phi = \pi/2$ or $\phi = 3\pi/2$. In both cases $\cos(\phi) = 0$ and, again by (2.13), Equation (I) follows. If conversely ρ fulfils (I), let $\varphi \in [0, 2\pi)$ be the rotation angle of $R(\rho)$. We deduce from Equation (I) that $k^2 - \lambda^2 - \mu^2 - \nu^2 = 0$ and one therefore has $\cos(\varphi) = 0$ by (2.13). Thus $\varphi = \pi/2$ or $\varphi = 3\pi/2$ which implies that $R(\rho)$ has order 4.

(4) Assume that $R(\rho)$ has order 6, then one easily deduces that the rotation angle is $\phi = \pi/3$ or $\phi = 5\pi/3$. Inserting $\cos(\phi) = 1/2$ into Equation (2.13) results in Equation (II). Now let in turn ρ satisfy Equation (II) and denote by $\varphi \in [0, 2\pi)$ the rotation angle of the corresponding rotation $R(\rho)$. By (2.13), one has $\cos(\varphi) = 1/2$ implying $\varphi = \pi/3$ or $\varphi = 5\pi/3$. Thus $R(\rho)$ has order 6.

(5) If $R(\rho)$ has order 3, then simple calculations show that $\phi = 2\pi/3$ or $\phi = 4\pi/3$ holds for the rotation angle ϕ. In both cases one has $\cos(\phi) = -1/2$ which, together with Equation (2.13), leads to Equation (III). Conversely, denoting by $\varphi \in [0, 2\pi)$ the rotation angle of $R(\rho)$ and letting ρ fulfil Equation (III), one obtains $\cos(\varphi) = -1/2$. This yields $\varphi = 2\pi/3$ or $\varphi = 4\pi/3$ and hence $R(\rho)$ has order 3.

□

We need some further preliminaries before addressing the main problem of classifying the generalised dihedral subgroups of $\mathrm{SO}(3, \mathbb{Q})$.

LEMMA 2.39. *Any natural number t is either the square of an integer or $\sqrt{t}$ is irrational.* □

LEMMA 2.40. *For any integer s, one has either $s^2 \equiv 0 \bmod 4$ or $s^2 \equiv 1 \bmod 4$ (depending on whether s is even or odd).* □

LEMMA 2.41. *For any primitive quaternion $\rho = (\kappa, \lambda, \mu, \nu)$ the following holds.*

(1) *If $\kappa = 0$, then at least one of λ, μ, ν is odd.*

(2) *If Equation* (I) *holds, then κ is odd. Moreover, exactly one of λ, μ, ν is odd as well.*

(3) *If Equation* (II) *or Equation* (III) *holds, then $\kappa, \lambda, \mu, \nu$ are all odd.*

PROOF. (1) This follows immediately from the fact that ρ is primitive.

(2) By Lemma 2.40, the squares of integers are congruent to 0 or to $1 \bmod 4$. If $\kappa^2 \equiv 0 \bmod 4$, then $\lambda^2 + \mu^2 + \nu^2 \equiv 0 \bmod 4$ and hence, $\lambda^2 \equiv \mu^2 \equiv \nu^2 \equiv 0 \bmod 4$.

This implies that all λ, μ, ν are even. But κ is even as well, contradicting the fact that ρ is primitive. Thus $\kappa^2 \equiv 1 \bmod 4$, which shows that κ is odd. Then one furthermore has $1 \equiv \lambda^2+\mu^2+\nu^2 \bmod 4$. This implies that exactly one of λ^2, μ^2, ν^2 is congruent to $1 \bmod 4$, and therefore exactly one of λ, μ, ν is odd.

(3) Let Equation (III) hold. By Lemma 2.40, one has $\kappa^2 \equiv 0 \bmod 4$ or $\kappa^2 \equiv 1 \bmod 4$. In the first case one has $0 \equiv \lambda^2 + \mu^2 + \nu^2 \bmod 4$. Again by Lemma 2.40, all of λ^2, μ^2, ν^2 are congruent to 0. Thus $\kappa, \lambda, \mu, \nu$ are all even, which is a contradiction to ρ being primitive. Therefore, one has $\kappa^2 \equiv 1 \bmod 4$, which immediately yields $3 \equiv \lambda^2+\mu^2+\nu^2 \bmod 4$. Hence, $\lambda^2 \equiv \mu^2 \equiv \nu^2 \equiv 1 \bmod 4$. This shows that $\kappa, \lambda, \mu, \nu$ are all odd. The statement for Equation (II) now follows by Lemma 2.37. Namely, if $\kappa, \lambda, \mu, \nu$ form a solution of Equation (II), then $\kappa, 3\lambda, 3\mu, 3\nu$ form a solution of Equation (III). Thus they are all odd, implying that $\kappa, \lambda, \mu, \nu$ are odd.

□

2.6. Generalised dihedral subgroups of $\mathrm{SO}(3,\mathbb{Q})$

We have already encountered some examples of generalised dihedral subgroups of $\mathrm{SO}(3,\mathbb{Q})$. The cyclic group C_k, where $k \in \{1,2,3,4,6\}$, is a subgroup of $\mathrm{SO}(3,\mathbb{Q})$ generated by a rotation about $2\pi/k$. It can be viewed as a generalised dihedral group by adding to this generator a second redundant one, given by the rotation by 2π about an axis orthogonal to the rotation axis of the first generator. The dihedral groups D_r with $r \in \{1,2,3,4,6\}$ are also generalised dihedral subgroups of $\mathrm{SO}(3,\mathbb{Q})$. They are generated by a rotation of order r and one of order 2 whose rotation axes are separated by an orthogonal angle, cf. Section 2.3. Furthermore, it was shown that the rotation symmetry group of the cube, S_4, is the generalised dihedral group $G_{1/4}(4,4)$ (cf. p. 34). It is generated by the two rotations by $\pi/2$ about the x-axis and y-axis, respectively. In matrix form, with respect to the canonical basis, these two generators are

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{pmatrix}.$$

Therefore, S_4 is a generalised dihedral subgroup of $\mathrm{SO}(3,\mathbb{Q})$.

All these examples are finite groups. What about other generalised dihedral groups, which are in most cases free products or amalgamated free products of cyclic or dihedral groups? It turns out that, in the rational case, no infinite generalised dihedral group occurs.

THEOREM 2.42. *Let G be a nontrivial generalised dihedral subgroup of $\mathrm{SO}(3,\mathbb{Q})$, i.e., let G be a nontrivial subgroup of $\mathrm{SO}(3,\mathbb{Q})$ that is generated by two rotations of finite order about axes that are separated by an angle of finite order. Then G is one of the following groups*

the cyclic group C_k with $k \in \{2,3,4,6\}$,
the dihedral group D_ℓ of order 2ℓ with $\ell \in \{2,3,4,6\}$,

or the symmetric group S_4.

In particular, all generalised dihedral subgroups of $\mathrm{SO}(3,\mathbb{Q})$ *are finite.*

Due to its length, we give the proof of this theorem separately in the next section. Prior to the proof, let us formulate one consequence obtained by combining this result with the classification of finite subgroups of $\mathrm{SO}(3,\mathbb{Q})$ in Corollary 2.24.

COROLLARY 2.43. *The finite subgroups of* $\mathrm{SO}(3,\mathbb{Q})$ *are exactly the generalised dihedral subgroups of* $\mathrm{SO}(3,\mathbb{Q})$ *and the alternating group* A_4. $\square$

It would be interesting to know which of the finite groups of Theorem 2.42 themselves appear as groups of coincidence rotations of a lattice. This is work in progress, but it can be said that not all of them appear.

2.7. Proof of Theorem 2.42

The finite subgroups of $\mathrm{SO}(3,\mathbb{Q})$ are the ones stated in the theorem, except for the alternating group A_4 (cf. Corollary 2.24). By Lemma 2.26, A_4 can be generated by a pair of finite order rotations, but only with axes separated by an irrational angle. All the other groups from the list in the theorem are generalised dihedral groups. The cyclic group C_k is generated by a rotation of order k and a rotation of order 1. The dihedral group D_ℓ is generated by a rotation of order ℓ and one of order 2 about axes that are separated by an orthogonal angle. The rotation symmetry group of the cube S_4 is generated by two generators of order 4 with orthogonal rotation axes. It remains to show that these are the only generalised dihedral subgroups of $\mathrm{SO}(3,\mathbb{Q})$. We employ Cayley's parametrisation of $\mathrm{SO}(3,\mathbb{Q})$ with primitive quaternions.

Let $\rho=(\kappa,\lambda,\mu,\nu)$ and $\rho'=(\kappa',\lambda',\mu',\nu')$ be two primitive quaternions such that $R(\rho)$ and $R(\rho')$ generate G and such that $R(\rho)$ and $R(\rho')$ are rotations of finite order p and q, respectively. Let their rotation axes be separated by an angle of $2\pi n/m$ with $n,m\in\mathbb{N}\backslash\{0\}$, $n<m$, $\gcd(n,m)=1$ and $m>2$.

Without loss of generality, we assume that m is either odd or a multiple of 4. Namely, if m is an odd multiple of 2, say $m=2m'$ with m' odd, replace ρ' by $-\rho'$. This parametrises the same orthogonal matrix $R(\rho')=R(-\rho')$, but the rotation axis of $R(-\rho')$ is given by $-(\lambda',\mu',\nu')^t$ making an angle of $2\pi\big((m'+n)/2m'\big)$ with the rotation axis of $R(\rho)$. Since m' and n are odd, $m'+n=:2t$ is even, and thus $2\pi\big((m'+n)/2m'\big)=2\pi t/m'$. It is now straightforward to show that $\gcd(t,m')=1$ and we can thus replace m by m' and n by t.

Since $(\lambda,\mu,\nu)^t$ and $(\lambda',\mu',\nu')^t$ are the rotation axes of $R(\rho)$ and $R(\rho')$, one has

$$\cos\left(\frac{2\pi n}{m}\right)=\frac{\lambda\lambda'+\mu\mu'+\nu\nu'}{\sqrt{\lambda^2+\mu^2+\nu^2}\cdot\sqrt{(\lambda')^2+(\mu')^2+(\nu')^2}}\,. \tag{IV}$$

Write $G=:G_{n/m}(p,q)$. Due to Lemma 2.36, we have to check for $p,q\in\{1,2,3,4,6\}$. For $q=1$ (or $p=1$) G is finite, namely $G=C_q$ (or $G=C_p$, respectively). Of course, $G_{n/m}(p,q)=G_{n/m}(q,p)$.

If $p = q = 4$, $R(\rho)$ and $R(\rho')$ are both rotations of order 4. By Lemma 2.38(3), this leads to the following equations.

$$\kappa^2 = \lambda^2 + \mu^2 + \nu^2 \tag{I}$$

$$(\kappa')^2 = (\lambda')^2 + (\mu')^2 + (\nu')^2 \tag{I$'$}$$

Furthermore, one has

$$\cos\left(\frac{2\pi n}{m}\right) = \frac{\lambda\lambda' + \mu\mu' + \nu\nu'}{\sqrt{\lambda^2 + \mu^2 + \nu^2} \cdot \sqrt{(\lambda')^2 + (\mu')^2 + (\nu')^2}}. \tag{IV}$$

$G_{n/m}(4, 4)$ is a subgroup of SO(3, $\mathbb{Q}$) if and only if the three equations above have an integer solution $\{\kappa, \lambda, \mu, \nu, \kappa', \lambda', \mu', \nu'\}$ that is subject to the conditions $\gcd(\kappa, \lambda, \mu, \nu) = 1$ and $\gcd(\kappa', \lambda', \mu', \nu') = 1$; cf. Section 2.5.1.

If the rotation axes of $R(\rho)$ and $R(\rho')$ are orthogonal (i.e. if $m = 4$), G is the rotation symmetry group of the cube S_4 (cf. p. 34).

If $m \neq 4$, two cases are to be considered. One is where m is odd, the other where m is a multiple of 4. But first consider the equations at hand. By Lemma 2.41(2), Equations (I) and (I$'$) imply that κ and κ' are odd. Furthermore, inserting these two equations into Equation (IV) yields

$$\cos\left(\frac{2\pi n}{m}\right) = \frac{\lambda\lambda' + \mu\mu' + \nu\nu'}{\kappa\kappa'}. \tag{2.14}$$

If m is odd, Theorems 2.29(1) and 2.25(2) show that

$$G_{n/m}(4, 4) \simeq C_4 \star_{C_2} D_m,$$

where $C_4 \star_{C_2} D_m$ denotes the free product of D_m and C_4 with amalgamated subgroup C_2. This group contains D_m as a subgroup as seen in Section 2.1.3. Assuming that $G_{n/m}(4, 4)$ is a subgroup of SO(3, $\mathbb{Q}$), one has

$$D_m \subset G_{n/m}(4, 4) \subset \mathrm{SO}(3, \mathbb{Q}).$$

Now Corollary 2.24 yields $m \in \{1, 2, 3, 4, 6\}$ as a necessary condition. But m is odd and $m > 2$, hence $m = 3$. Using $\cos(2\pi n/3) = -1/2$ in Equation (2.14) yields

$$-\frac{1}{2} = \frac{\lambda\lambda' + \mu\mu' + \nu\nu'}{\kappa\kappa'},$$

and thus

$$-\kappa\kappa' = 2(\lambda\lambda' + \mu\mu' + \nu\nu').$$

The left hand side of this equation is odd, because κ and κ' are both odd. But the right hand side is even. This shows that the system of equations consisting of (I), (I$'$) and (IV) does not have a solution over the integers if m is odd. Hence $G_{n/m}(4, 4)$ is not a subgroup of SO(3, $\mathbb{Q}$) for m odd.

If m is divisible by 4 but not by 8, Theorem 2.29(2) is applicable and gives

$$G_{n/m}(4,4) \simeq G_{1/4}(4,m) \simeq D_{\mathrm{lcm}(4,m)} \star_{D_4} S_4,$$

where Theorem 2.25(4) was used in the last step. This group contains $D_{\mathrm{lcm}(4,m)}$ as a subgroup. Thus $\mathrm{lcm}(4,m) \in \{1,2,3,4,6\}$ by Corollary 2.24. Since 4 divides $\mathrm{lcm}(m,4)$, this leaves only $\mathrm{lcm}(m,4) = 4$ which is impossible due to $m \neq 4$.

If m is divisible by 8, Theorem 2.30(2) yields

$$G_{n/m}(4,4) \simeq G_{1/4}(r,4) \star_{D_r} G_{1/4}(r,4),$$

where $r = \mathrm{lcm}(4, m/2)$. Hence one has $4 \mid r$. The group $G_{(1/4)}(r,4) \star_{D_r} G_{1/4}(r,4)$ contains D_r as a subgroup. By Corollary 2.24, this restricts the possible values of r to the set $\{1,2,3,4,6\}$. Thus $r = 4$, implying $m = 8$ (because $m \neq 4$ and $4 \mid m$). For $m = 8$ one has $\cos(2\pi n/m) = \pm\sqrt{2}/2$. But then Equation (2.14) yields

$$\pm\sqrt{2} = \frac{2(\lambda\lambda' + \mu\mu' + \nu\nu')}{\kappa\kappa'} \in \mathbb{Q},$$

which is a contradiction. Hence $G_{n/m}(4,4)$ is not a subgroup of $\mathrm{SO}(3,\mathbb{Q})$ for $m \neq 4$.

The next case that we consider is where $p = 4$ and $q = 2$. Then $R(\rho)$ is a rotation of order 4 and $R(\rho')$ is a rotation of order 2. By Lemma 2.38, this leads to the following equations.

$$(0') \qquad \kappa' = 0$$

$$(\mathrm{I}) \qquad \kappa^2 = \lambda^2 + \mu^2 + \nu^2$$

$$(\mathrm{IV}) \qquad \cos\left(\frac{2\pi n}{m}\right) = \frac{\lambda\lambda' + \mu\mu' + \nu\nu'}{\sqrt{\lambda^2+\mu^2+\nu^2}\cdot\sqrt{(\lambda')^2+(\mu')^2+(\nu')^2}}$$

By Lemma 2.41(2), κ is odd. Inserting Equation (I) into Equation (IV) gives

$$(2.15) \qquad \cos\left(\frac{2\pi n}{m}\right) = \frac{\lambda\lambda' + \mu\mu' + \nu\nu'}{\kappa\sqrt{(\lambda')^2+(\mu')^2+(\nu')^2}}.$$

If the rotation axes of $R(\rho)$ and $R(\rho')$ are orthogonal, G is the dihedral group D_4 of order 8 (see p. 34).

If $m \neq 4$, there are again two cases. One is where m is odd, the other where m is divisible by 4. If m is odd, Theorem 2.29(1) and Theorem 2.25(2) immediately imply that

$$G_{n/m}(4,2) \simeq C_4 \star_{C_2} D_m.$$

This group contains D_m as a subgroup. This restricts the possible values of m to the set $\{1,2,3,4,6\}$, and because $m > 2$, $m = 3$ follows. Inserting $\cos(2\pi n/3) = -1/2$ in (2.15) yields

$$(2.16) \qquad -\kappa\sqrt{(\lambda')^2+(\mu')^2+(\nu')^2} = 2(\lambda\lambda' + \mu\mu' + \nu\nu').$$

Due to Lemma 2.39, $\sqrt{(\lambda')^2+(\mu')^2+(\nu')^2}$ is either irrational or an integer. If it is irrational, Equation (2.16) has no solution over the integers. If $\sqrt{(\lambda')^2+(\mu')^2+(\nu')^2} = h \in \mathbb{Z}$, Equation (2.16) reads

$$-\kappa h = 2(\lambda\lambda' + \mu\mu' + \nu\nu').$$

The right hand side is even and κ is odd, so h is even, say $h = 2h'$. Thus, one has $(\lambda')^2+(\mu')^2+(\nu')^2 = 4(h')^2 \equiv 0 \bmod 4$. Hence $(\lambda')^2 \equiv (\mu')^2 \equiv (\nu')^2 \equiv 0 \bmod 4$, because they are squares. This implies that $(\lambda')^2, (\mu')^2, (\nu')^2$ are even and so are λ', μ', ν'. But $\kappa' = 0$ is also even, which is a contradiction to ρ' being primitive.

If m is divisible by 4 and by 8, Theorem 2.30(3) states that

$$G_{n/m}(4,2) \simeq G_{1/4}(s,4) \star_{D_2} D_2 \simeq G_{1/4}(s,4),$$

where $s = \operatorname{lcm}(4, m/2)$ and where we used Lemma 2.18 in the last step. We use the fact that s is divisible by 4 in Theorem 2.25(4) to obtain

$$G_{n/m}(4,2) \simeq G_{1/4}(s,4) \simeq D_s \star_{D_4} S_4.$$

Since D_s is a subgroup of $D_s \star_{D_4} S_4$, only $s = 4$ is possible by a similar argument as used before. Hence $m = 8$, and due to Lemma 2.18, one has

$$G_{n/8}(4,2) \simeq D_4 \star_{D_4} S_4 \simeq S_4.$$

If m is divisible by 4 but not divisible by 8, Theorem 2.30(4) states

$$G_{n/m}(4,2) \simeq D_4 \star_{D_2} D_t,$$

where $t = \operatorname{lcm}(2, m/2)$. Hence, t is even and $t > 2$. Since D_t is a subgroup of $D_4 \star_{D_2} D_t$, this leaves $t \in \{4, 6\}$. If $t = 4$, then $m = 8$, which contradicts $8 \nmid m$. If $t = 6$, one has $m = 12$ (because $m/2$ is even and excludes $m = 6$) and thus,

$$G_{n/12}(4,2) \simeq D_4 \star_{D_2} D_6.$$

Inserting $\cos(2\pi n/12) = \pm\sqrt{3}/2$ into Equation (2.15) yields

$$\pm\sqrt{3}\sqrt{(\lambda')^2+(\mu')^2+(\nu')^2} = \frac{2(\lambda\lambda'+\mu\mu'+\nu\nu')}{\kappa} \in \mathbb{Q}.$$

By Lemma 2.39, $3\big((\lambda')^2+(\mu')^2+(\nu')^2\big)$ is a square number, say r^2 with $r \in \mathbb{Z}$. Then

$$\pm\kappa r = 2(\lambda\lambda'+\mu\mu'+\nu\nu')$$

is an even integer. Since κ is odd, $r = 2r'$ is even. From $3\big((\lambda')^2+(\mu')^2+(\nu')^2\big) = 4(r')^2$ follows $3 \mid r'$, because 3 is a prime number. Hence $3\big((\lambda')^2+(\mu')^2+(\nu')^2\big) = 4 \cdot 9(r'')^2$ for some $r'' \in \mathbb{Z}$. Thus, one has

$$(\lambda')^2+(\mu')^2+(\nu')^2 = 4 \cdot 3(r'')^2,$$

which shows that the left hand side is divisible by 4. As before, this leads to a contradiction, because ρ' is primitive. This completes the case where $p = 4$ and $q = 2$.

Since rotations of order 3 and 6 lead to the equivalent Equations (II) and (III) (cf. Lemma 2.37), we can combine certain cases.

The cases where $p = 4$ and $q = 3$ or where $p = 4$ and $q = 6$ can be treated simultaneously, because both lead to the system of equations

$$\kappa^2 = \lambda^2 + \mu^2 + \nu^2 \tag{I}$$

$$3(\kappa')^2 = (\lambda')^2 + (\mu')^2 + (\nu')^2 \tag{III$'$}$$

$$\cos\left(\frac{2\pi n}{m}\right) = \frac{\lambda\lambda' + \mu\mu' + \nu\nu'}{\sqrt{\lambda^2 + \mu^2 + \nu^2} \cdot \sqrt{(\lambda')^2 + (\mu')^2 + (\nu')^2}}. \tag{IV}$$

If the rotation axes of $R(\rho)$ and $R(\rho')$ are separated by an orthogonal angle, Equation (IV) is

$$0 = \lambda\lambda' + \mu\mu' + \nu\nu'. \tag{2.17}$$

By Lemma 2.41(3), Equation (III$'$) shows that $\kappa', \lambda', \mu', \nu'$ are all odd. Further, due to Lemma 2.41(2), Equation (I) implies that κ is odd and exactly one of λ, μ, ν is also odd. This shows that $\lambda\lambda' + \mu\mu' + \nu\nu'$ is odd, which is a contradiction to (2.17). Thus neither $G_{n/4}(4,3)$ nor $G_{n/4}(4,6)$ is a subgroup of $\mathrm{SO}(3,\mathbb{Q})$.

If $m \neq 4$, insert Equations (I) and (III$'$) into (IV) to obtain

$$\cos\left(\frac{2\pi n}{m}\right) = \frac{\lambda\lambda' + \mu\mu' + \nu\nu'}{\kappa\kappa'\sqrt{3}}. \tag{2.18}$$

Hence $\cos(2\pi n/m) \notin \mathbb{Q}$, but $\cos(2\pi n/m) \in \mathbb{Q}(\sqrt{3})$. This implies

$$\mathbb{Q}\Big(2\cos\Big(\frac{2\pi n}{m}\Big)\Big) = \mathbb{Q}\Big(\cos\Big(\frac{2\pi n}{m}\Big)\Big) = \mathbb{Q}(\sqrt{3}). \tag{2.19}$$

Furthermore, $\mathbb{Q}\left(2\cos(2\pi/m)\right) = \mathbb{Q}\left(\zeta_m + \overline{\zeta_m}\right)$ is the maximal real subfield of the cyclotomic field $\mathbb{Q}(\zeta_m)$. Note that $\mathbb{Q}(\zeta_m) = \mathbb{Q}(\zeta_m^n)$, because ζ_m^n is also a primitive m-th root of unity since m and n are relatively prime. This implies $\mathbb{Q}(2\cos(2\pi n/m)) = \mathbb{Q}(2\cos(2\pi/m))$. Moreover, the field $\mathbb{Q}(2\cos(2\pi n/m))$ has degree $\varphi(m)/2$ over $\mathbb{Q}$ by Corollary 2.3. On the other hand, $\mathbb{Q}(2\cos(2\pi n/m))$ coincides with $\mathbb{Q}(\sqrt{3})$ and therefore has degree 2 over $\mathbb{Q}$. Hence $2 = \varphi(m)/2$, leaving $m \in \{5, 8, 10, 12\}$. But m is odd or a multiple of 4, which leaves $m \in \{5, 8, 12\}$. Note that one has $\sqrt{3} = 2\cos(2\pi/12)$ and therefore $\mathbb{Q}(\sqrt{3}) = \mathbb{Q}(2\cos(2\pi/12))$.

For $m = 5$, Equation (2.19) then leads to $\mathbb{Q}(2\cos(2\pi/5)) = \mathbb{Q}(2\cos(2\pi/12))$, contradicting Corollary 2.5.

If $m = 8$, use $2\cos(2\pi n/8) = \pm\sqrt{2}$ in Equation (2.19) to get $\mathbb{Q}(\sqrt{2}) = \mathbb{Q}(\sqrt{3})$, which is nonsense.

This leaves $m = 12$. Inserting $\cos(2\pi n/12) = \pm\sqrt{3}/2$ into Equation (2.18) gives

$$\pm\frac{\sqrt{3}}{2} = \frac{\lambda\lambda' + \mu\mu' + \nu\nu'}{\kappa\kappa'\sqrt{3}},$$

and thus,

$$\pm 3\kappa\kappa' = 2(\lambda\lambda' + \mu\mu' + \nu\nu'). \tag{2.20}$$

The left hand side is odd, because κ and κ' are odd. But the right hand side is even, which is a contradiction. Therefore $G_{n/m}(4,3)$ and $G_{n/m}(4,6)$ are not subgroups of $\mathrm{SO}(3,\mathbb{Q})$.

If $p = 6$ and $q = 2$ or if $p = 3$ and $q = 2$, one has the following system of equations.

$$\kappa' = 0 \tag{0'}$$

$$3\kappa^2 = \lambda^2 + \mu^2 + \nu^2 \tag{III}$$

$$\cos\left(\frac{2\pi n}{m}\right) = \frac{\lambda\lambda' + \mu\mu' + \nu\nu'}{\sqrt{\lambda^2+\mu^2+\nu^2}\cdot\sqrt{(\lambda')^2+(\mu')^2+(\nu')^2}} \tag{IV}$$

If $m = 4$, G is the dihedral group D_6 of order 12 (if $p = 6, q = 2$), or it is the dihedral group D_3 of order 6 (if $p = 3, q = 2$).

Otherwise, note that Lemma 2.41(3) implies that $\kappa, \lambda, \mu, \nu$ are all odd, and insert Equation (III) into (IV) to get

$$\cos\left(\frac{2\pi n}{m}\right) = \frac{\lambda\lambda' + \mu\mu' + \nu\nu'}{\kappa\cdot\sqrt{3}\cdot\sqrt{(\lambda')^2+(\mu')^2+(\nu')^2}}\,. \tag{2.21}$$

There are now two cases to consider depending on whether $l := \sqrt{3\big((\lambda')^2+(\mu')^2+(\nu')^2\big)}$ is irrational or an integer.

If $l \in \mathbb{Z}$, then one has $\cos(2\pi n/m) \in \mathbb{Q}$ by Equation (2.21). Similar to the argumentation above, we deduce $\mathbb{Q} = \mathbb{Q}(2\cos(2\pi n/m))$ (see also the proof of Lemma 2.36). Hence the degree $\varphi(m)/2$ of $\mathbb{Q}(2\cos(2\pi n/m))$ over $\mathbb{Q}$ is equal to 1, implying $\varphi(m) = 2$ and thus $m \in \{3,4,6\}$. Either m is odd or a multiple of 4 (but $m \neq 4$), which leaves only $m = 3$. Inserting $\cos(2\pi n/3) = -1/2$ into Equation (2.21) yields

$$-l\kappa = 2(\lambda\lambda' + \mu\mu' + \nu\nu'). \tag{2.22}$$

Thus l is even, because κ is odd. But on the other hand, one has $l^2 = 3\big((\lambda')^2+(\mu')^2+(\nu')^2\big)$. By Lemma 2.41(3) l is odd, which is a contradiction. The case that is left is where l is irrational. By Equation (2.21) one has $\cos(2\pi n/m) \notin \mathbb{Q}$, but $\cos(2\pi n/m) \in \mathbb{Q}(l)$. A similar argument as previously shows $\mathbb{Q}(2\cos(2\pi n/m)) = \mathbb{Q}(l)$ and hence $\varphi(m) = 4$, implying $m \in \{5,8,10,12\}$. Since m is odd or a multiple of 4, $m \in \{5,8,12\}$. For $m = 5$, one has $\cos(2\pi n/5) = \pm(\sqrt{5}\mp 1)/4$ and hence Equation (2.21) yields

$$\pm\frac{1}{4}(\sqrt{5}\mp 1)\cdot\sqrt{3\big((\lambda')^2+(\mu')^2+(\nu')^2\big)} \in \mathbb{Q}.$$

Then its square $\frac{1}{8}(3\mp\sqrt{5})\cdot 3\big((\lambda')^2+(\mu')^2+(\nu')^2\big)$ is also rational. But this implies that $\sqrt{5}$ is rational, which is nonsense.

For $m = 8$ we insert $\cos(2\pi n/8) = \pm\sqrt{2}/2$ into Equation (2.21) to get

$$\pm\sqrt{6\big((\lambda')^2 + (\mu')^2 + (\nu')^2\big)} = \frac{2(\lambda\lambda' + \mu\mu' + \nu\nu')}{\kappa} \in \mathbb{Q}. \tag{2.23}$$

By Lemma 2.39, $6\big((\lambda')^2 + (\mu')^2 + (\nu')^2\big)$ is the square of an integer s and therefore, it is congruent to 0 or 1 mod 4. Considering the square of (2.23), then it easily follows that $6\big((\lambda')^2 + (\mu')^2 + (\nu')^2\big) \equiv 0 \bmod 4$, because κ is odd. Thus exactly one of $(\lambda')^2, (\mu')^2, (\nu')^2$ is congruent to 0 mod 4, which implies that exactly one of λ', μ', ν' is even and so $\lambda\lambda' + \mu\mu' + \nu\nu'$ is even as well (because λ, μ, ν are odd). Thus $(\lambda\lambda' + \mu\mu' + \nu\nu')^2 \equiv 0 \bmod 4$. Consider again the square of Equation (2.23). Then one has

$$3\kappa^2\big((\lambda')^2 + (\mu')^2 + (\nu')^2\big) = 2\left(\lambda\lambda' + \mu\mu' + \nu\nu'\right)^2.$$

Since κ is odd, one has $\kappa^2 \equiv 1 \bmod 4$. Therefore, the equation above yields $6 \equiv 0 \bmod 4$, which clearly is a contradiction.

For $m = 12$, $\cos(2\pi n/12) = \pm\sqrt{3}/2$ yields

$$\pm\sqrt{(\lambda')^2 + (\mu')^2 + (\nu')^2} = \frac{2(\lambda\lambda' + \mu\mu' + \nu\nu')}{3\kappa} \in \mathbb{Q}$$

by Equation (2.21). This implies that $(\lambda')^2 + (\mu')^2 + (\nu')^2$ is the square of an integer t by Lemma 2.39. One has

$$\pm 3\kappa t = 2(\lambda\lambda' + \mu\mu' + \nu\nu'),$$

and hence t is even, because κ is odd. Then $(\lambda')^2 + (\mu')^2 + (\nu')^2 = t^2 \equiv 0 \bmod 4$, and thus, $(\lambda')^2 \equiv (\mu')^2 \equiv (\nu')^2 \equiv 0 \bmod 4$, showing that λ', μ', ν' are all even. But so is κ', which is a contradiction to ρ' being primitive. To sum up, $G_{n/m}(6,2)$ and $G_{n/m}(3,2)$ are not subgroups of $\mathrm{SO}(3,\mathbb{Q})$ for $m \neq 4$.

If $p = q = 6$, or if $p = q = 3$, or if $p = 6$ and $q = 3$, one has the following system of equations.

$$3\kappa^2 = \lambda^2 + \mu^2 + \nu^2 \tag{III}$$

$$3(\kappa')^2 = (\lambda')^2 + (\mu')^2 + (\nu')^2 \tag{III$'$}$$

$$\cos\left(\frac{2\pi n}{m}\right) = \frac{\lambda\lambda' + \mu\mu' + \nu\nu'}{\sqrt{\lambda^2 + \mu^2 + \nu^2} \cdot \sqrt{(\lambda')^2 + (\mu')^2 + (\nu')^2}} \tag{IV}$$

By Lemma 2.41(3), one deduces from Equations (III) and (III$'$) that $\kappa, \lambda, \mu, \nu$ as well as $\kappa', \lambda', \mu', \nu'$ are all odd.

If $m = 4$, Equation (IV) reads

$$0 = \lambda\lambda' + \mu\mu' + \nu\nu'.$$

The left hand side is even, while the right hand side is odd. This is a contradiction.

If $m \neq 4$, we insert Equations (III) and (III′) into (IV) to obtain

$$\cos\left(\frac{2\pi n}{m}\right) = \frac{\lambda\lambda' + \mu\mu' + \nu\nu'}{3\kappa\kappa'} \in \mathbb{Q}. \tag{2.24}$$

By a similar argument as before, one receives $\mathbb{Q}(2\cos(2\pi n/m)) = \mathbb{Q}$, implying $\varphi(m) = 2$ and hence $m \in \{3, 4, 6\}$. Since m is odd or m is divisible by 4 (but $m \neq 4$), this leaves only $m = 3$. Using $\cos(2\pi n/3) = -1/2$ in Equation (2.24) yields

$$-3\kappa\kappa' = 2(\lambda\lambda' + \mu\mu' + \nu\nu').$$

The right hand side of this equation is even, whereas the left hand side is not. This is a contradiction. Therefore, neither $G_{n/m}(6,6)$ nor $G_{n/m}(6,3)$ nor $G_{n/m}(3,3)$ is a subgroup of SO(3, $\mathbb{Q}$).

The only case that remains is where $p = q = 2$. This leads to the following equations.

$$\kappa = 0 \tag{0}$$

$$\kappa' = 0 \tag{0′}$$

$$\cos\left(\frac{2\pi n}{m}\right) = \frac{\lambda\lambda' + \mu\mu' + \nu\nu'}{\sqrt{\lambda^2 + \mu^2 + \nu^2} \cdot \sqrt{(\lambda')^2 + (\mu')^2 + (\nu')^2}} \tag{IV}$$

If the rotation axes of $R(\rho)$ and $R(\rho')$ are separated by an orthogonal angle, one has $G_{1/4}(2,2) = D_2$ as seen on p. 34.

If $m \neq 4$, consider $a := (\lambda^2 + \mu^2 + \nu^2) \cdot ((\lambda')^2 + (\mu')^2 + (\nu')^2)$. If a is the square of an integer r, then Equation (IV) yields

$$\cos\left(\frac{2\pi n}{m}\right) = \frac{\lambda\lambda' + \mu\mu' + \nu\nu'}{r} \in \mathbb{Q}.$$

By a similar argument as before, this implies $m = 3$. Now Theorems 2.29(1) and 2.25(2) state that

$$G_{n/3}(2,2) \simeq G_{1/4}(2,3) \simeq D_3.$$

By Lemma 2.39, the only case that is left is where $\sqrt{a}$ is irrational. Then one has $\mathbb{Q}(2\cos(2\pi n/m)) = \mathbb{Q}(\sqrt{a})$. As previously, one deduces that $m \in \{5, 8, 10, 12\}$ and since m is odd or a multiple of 4, one has $m \in \{5, 8, 12\}$.

For $m = 5$ Theorems 2.29(1) and 2.25(2) yield

$$G_{n/5}(2,2) \simeq G_{1/4}(2,5) \simeq D_5.$$

For $m = 8$, Theorem 2.30(1) states that, for all $n < 8$ with $\gcd(n, 8) = 1$, the group $G_{n/8}(2,2)$ has the group presentation

$$G_{n/8}(2,2) = \langle \alpha, \beta, \gamma \mid \alpha^2, \beta^2, \gamma^2, (\alpha\beta)^2, (\gamma\beta)^2, (\alpha\gamma)^2\beta \rangle.$$

By Example 2.16(2) this is a quotient of $D_2 \star_{C_2} D_2$ by the last relation $(\alpha\gamma)^2\beta$. More precisely, since $\beta = (\alpha\gamma)^2$ holds in $G_{n/8}(2,2)$, we use Theorem 2.9 to obtain

$$\begin{aligned} G_{n/8}(2,2) &= \langle \alpha, \gamma \mid \alpha^2, (\alpha\gamma)^4, \gamma^2, (\alpha(\alpha\gamma)^2)^2, (\gamma(\alpha\gamma)^2)^2 \rangle \\ &= \langle \alpha, \gamma \mid \alpha^2, (\alpha\gamma)^4, \gamma^2 \rangle, \end{aligned}$$

because the two relators $(\alpha(\alpha\gamma)^2)^2$ and $(\gamma(\alpha\gamma)^2)^2$ are redundant. Set $\nu := \alpha\gamma$. Then one has

$$\begin{aligned} G_{n/8}(2,2) &= \langle \alpha, \gamma, \nu \mid \alpha^2, \nu^4, \gamma^2 \rangle \\ &= \langle \gamma, \nu \mid \nu^4, \gamma^2, (\nu\gamma)^2 \rangle \\ &= D_4. \end{aligned}$$

For $m = 12$, Theorem 2.30(1) implies

$$G_{n/12}(2,2) = \langle \alpha, \beta, \gamma \mid \alpha^2, \beta^2, \gamma^2, (\alpha\beta)^2, (\gamma\beta)^2, (\alpha\gamma)^3\beta \rangle$$

for all $n < 12$ with $\gcd(n, 12) = 1$. By Example 2.16(2) this is a quotient of $D_2 \star_{C_2} D_2$ by the last relation $(\alpha\gamma)^3\beta$. To specify, we use Theorem 2.9 and get

$$\begin{aligned} G_{n/12}(2,2) &= \langle \alpha, \gamma \mid \alpha^2, (\alpha\gamma)^6, \gamma^2, (\alpha(\alpha\gamma)^3)^2, (\gamma(\alpha\gamma)^3)^2 \rangle \\ &= \langle \alpha, \gamma \mid \alpha^2, (\alpha\gamma)^6, \gamma^2 \rangle, \end{aligned}$$

because the two relators $(\alpha(\alpha\gamma)^3)^2$ and $(\gamma(\alpha\gamma)^3)^2$ are superfluous. Setting $\nu = \alpha\gamma$ yields

$$\begin{aligned} G_{n/12}(2,2) &= \langle \gamma, \nu \mid \nu^6, \gamma^2, (\nu\gamma)^2 \rangle \\ &= D_6. \end{aligned}$$

This completes the proof. □

REMARK 2.44. In the proof above, we used the fact that $\mathrm{tr}(R(\rho)) \in \mathbb{Q}$ repeatedly via Equation (2.13). Apart from $\det(R(\rho))$ and $\mathrm{tr}(R(\rho))$, there is another invariant that could be useful for our calculations. This is the second elementary polynomial that appears as the coefficient of x in the expanded characteristic polynomial $\chi(x)$ of $R(\rho)$. But this turns out to be $-\mathrm{tr}(R(\rho))$ and thus yields no further information.

2.8. Generalised dihedral subgroups of $\mathrm{SO}(3, \mathbb{Q}(\tau))$

Concerning coincidence isometries in 3-space, another important example besides the cubic lattice $\mathbb{Z}^3$ is given by the standard icosahedral modules, i.e. by the face centred icosahedral module $\mathcal{M}_{\mathcal{F}}$, the body centred icosahedral module $\mathcal{M}_{\mathcal{B}}$ and the primitive icosahedral module $\mathcal{M}_{\mathcal{P}}$. They are defined as follows.

$$\begin{aligned} \mathcal{M}_{\mathcal{B}} &= \left\{ \sum\nolimits_{i=1}^{3} a_i e_i \;\middle|\; a_i \in \mathbb{Z}[\tau] \text{ and } \tau^2 a_1 + \tau a_2 + a_3 \equiv 0 \pmod 2 \right\} \\ \mathcal{M}_{\mathcal{P}} &= \{ x \in \mathcal{M}_{\mathcal{B}} \mid a_1 + a_2 + a_3 \equiv 0 \text{ or } \tau \pmod 2 \} \\ \mathcal{M}_{\mathcal{F}} &= \{ x \in \mathcal{M}_{\mathcal{B}} \mid a_1 + a_2 + a_3 \equiv 0 \pmod 2 \} \end{aligned}$$

These are, up to similarity, the only modules of rank 6 over $\mathbb{Z}$ with icosahedral symmetry, i.e., which are invariant under the icosahedral group $\mathcal{I}$ [**31**]. Note that $\mathcal{M}_{\mathcal{F}}$ and $\mathcal{M}_{\mathcal{B}}$ are $\mathbb{Z}[\tau]$-modules of rank 3 in $\mathbb{R}^3$. By Example 1.27(3), they are $\mathbb{Z}[\tau]$-modules over $\mathbb{Q}(\tau)$ which implies that their respective groups of coincidence and similarity isometries coincide; cf. Theorem 1.28. $\mathcal{M}_{\mathcal{P}}$ however fails to be a $\mathbb{Z}[\tau]$-module, but is a $\mathbb{Z}[2\tau]$-module only. The group of coincidence rotations is the same for all three modules, namely $\mathrm{SO}(3, \mathbb{Q}(\tau))$, where $\tau = (1+\sqrt{5})/2$ is the golden ratio (cf. [**2**, Pop. 5.3]).

Hence $\mathrm{SO}(3, \mathbb{Q}(\tau))$ and its subgroups are of natural interest. As seen in Section 2.3, one of these subgroups is the rotation symmetry group of the icosahedron. Additionally, all subgroups of $\mathrm{SO}(3, \mathbb{Q})$ are certainly also subgroups of $\mathrm{SO}(3, \mathbb{Q}(\tau))$. We recall some facts on the involved field $\mathbb{Q}(\tau)$ before stating all finite subgroups of $\mathrm{SO}(3, \mathbb{Q}(\tau))$ and classifying its finite generalised dihedral subgroups.

$\mathbb{Q}(\tau)$ is a quadratic real algebraic number field due to the identity $\tau^2 - \tau - 1 = 0$. Its ring of integers $\mathcal{O}_{\mathbb{Q}(\tau)} = \mathbb{Z}[\tau]$ is a unique factorisation domain ([**20**, Thm. 2.22]). Being a Dedekind domain, $\mathcal{O}_{\mathbb{Q}(\tau)}$ is hence also a principal ideal domain; cf. [**28**, Thms. 3.12 and 3.32]. This implies that $\mathbb{Q}(\tau)$ has class number one. By Theorem 2.35, $\mathrm{SO}(3, \mathbb{Q}(\tau))$ is thus parametrised by the set of $\mathbb{Z}[\tau]$-primitive quaternions via Cayley's parametrisation. Moreover, two $\mathbb{Z}[\tau]$-primitive quaternions ρ and ρ' yield the same matrix in $\mathrm{SO}(3, \mathbb{Q}(\tau))$ if and only if $\rho' = \varepsilon\rho$ for some unit ε of $\mathbb{Z}[\tau]$, i.e. for some $\varepsilon = \pm\tau^n$ with $n \in \mathbb{Z}$; cf. [**20**] for more on this.

Lemma 2.45. *Let R be an element of* $\mathrm{SO}(3, \mathbb{Q}(\tau))$ *of finite order. Then R has order* 1, 2, 3, 4, 5, 6 *or* 10.

Proof. We proceed similarly as in Lemma 2.36, making use of Cayley's parametrisation of $\mathrm{SO}(3, \mathbb{Q}(\tau))$ with $\mathbb{Z}[\tau]$-primitive quaternions. Let $\rho = (\kappa, \lambda, \mu, \nu) \in \mathbb{H}(\mathbb{Q}(\tau))$ be a $\mathbb{Z}[\tau]$-primitive quaternion such that $R(\rho) = R$. Since R has finite order, its rotation angle $\phi \in [0, 2\pi)$ can be written as $\phi = 2\pi n/m$, where $\gcd(n, m) = 1$. Equation (2.13) implies

$$\cos(\phi) = \frac{\kappa^2 - \lambda^2 - \mu^2 - \nu^2}{\kappa^2 + \lambda^2 + \mu^2 + \nu^2} \in \mathbb{Q}(\tau). \tag{2.25}$$

Thus $\mathbb{Q}(\cos(\phi)) = \mathbb{Q}$ or $\mathbb{Q}(\cos(\phi)) = \mathbb{Q}(\tau)$. Note that $\mathbb{Q}(\cos(\phi)) = \mathbb{Q}(2\cos(2\pi/m))$ is the maximal real subfield of $\mathbb{Q}(\zeta_m)$. For $m \geqslant 3$, one has $\varphi(m)/2 = 1$ in the first, and $\varphi(m)/2 = [\mathbb{Q}(\tau) : \mathbb{Q}] = 2$ in the second case due to Corollary 2.3. Hence $m \in \{3, 4, 6\}$ or $m \in \{5, 8, 10, 12\}$. The cases $m = 8$ and $m = 12$ are excluded, because $\sqrt{2}/2 = \cos(2\pi/8)$ and $\sqrt{3}/2 = \cos(2\pi/12)$ contradict (2.25). □

Combining this result with Theorem 2.23 immediately implies that the only possible finite subgroups of $\mathrm{SO}(3, \mathbb{Q}(\tau))$ are the groups C_n with $n \in \{1, 2, 3, 4, 5, 6, 10\}$, D_m with $m \in \{2, 3, 4, 5, 6, 10\}$, A_4, S_4 and A_5.

Now consider the following equations, some of which we have encountered before.

$$\rho = (0, \lambda, \mu, \nu) \tag{0}$$

$$\kappa^2 = \lambda^2 + \mu^2 + \nu^2 \tag{I}$$

$$\kappa^2 = 3(\lambda^2 + \mu^2 + \nu^2) \tag{II}$$

$$3\kappa^2 = \lambda^2 + \mu^2 + \nu^2 \tag{III}$$

$$(7 - 4\tau)\kappa^2 = \lambda^2 + \mu^2 + \nu^2 \tag{V}$$

$$5\kappa^2 = (7 - 4\tau)(\lambda^2 + \mu^2 + \nu^2) \tag{VI}$$

$$(7 - 4\tau)\kappa^2 = 5(\lambda^2 + \mu^2 + \nu^2) \tag{VII}$$

$$\kappa^2 = (7 - 4\tau)(\lambda^2 + \mu^2 + \nu^2) \tag{VIII}$$

Lemma 2.46.
(1) *Equations* (II) *and* (III) *are equivalent over* $\mathbb{Z}[\tau]$*, i.e., Equation* (II) *has a solution over* $\mathbb{Z}[\tau]$ *if and only if Equation* (III) *does.*
(2) *Equations* (V) *to* (VIII) *are equivalent over* $\mathbb{Z}[\tau]$*.*

Proof. The first statement follows analogously as in Lemma 2.37. In order to prove the second claim, we show (V)⇒(VII)⇒(VIII)⇒(VI)⇒(V). Let $\kappa, \lambda, \mu, \nu \in \mathbb{Z}[\tau]$ be a solution of (V). Then $(2\tau - 1)\kappa, \lambda, \mu, \nu \in \mathbb{Z}[\tau]$ satisfy (VII). If $\kappa, \lambda, \mu, \nu \in \mathbb{Z}[\tau]$ fulfil (VII), then $(7 - 4\tau)\kappa, (2\tau - 1)\lambda, (2\tau - 1)\mu, (2\tau - 1)\nu \in \mathbb{Z}[\tau]$ satisfy (VIII). Here, we make use of the fact that $5 = (2\tau - 1)^2$ is a square in $\mathbb{Z}[\tau]$. Given a solution $\kappa, \lambda, \mu, \nu \in \mathbb{Z}[\tau]$ of (VIII), the elements $\kappa, (2\tau - 1)\lambda, (2\tau - 1)\mu, (2\tau - 1)\nu \in \mathbb{Z}[\tau]$ form a solution of (VI). Finally, any solution $\kappa, \lambda, \mu, \nu \in \mathbb{Z}[\tau]$ of (VI) yields a solution of (V) that is contained in $\mathbb{Z}[\tau]$, namely $(2\tau - 1)\kappa, (7 - 4\tau)\lambda, (7 - 4\tau)\mu$ and $(7 - 4\tau)\nu$. □

Lemma 2.47. *If* $\rho = (\kappa, \lambda, \mu, \nu) \in \mathbb{H}(\mathbb{Q}(\tau))$ *is a* $\mathbb{Z}[\tau]$*-primitive quaternion, then the following statements hold for the rotation* $R(\rho) \in \mathrm{SO}(3, \mathbb{Q}(\tau))$*.*

(1) $R(\rho)$ *has order* 1 *if and only if* $\rho = (\varepsilon, 0, 0, 0)$ *where* ε *is a unit of* $\mathbb{Z}[\tau]$*.*
(2) $R(\rho)$ *has order* 2 *if and only if* $\kappa = 0$*.*
(3) $R(\rho)$ *has order* 4 *if and only if* ρ *satisfies Equation* (I)*.*
(4) $R(\rho)$ *has order* 6 *if and only if* ρ *satisfies Equation* (II)*.*
(5) $R(\rho)$ *has order* 3 *if and only if* ρ *satisfies Equation* (III)*.*
(6) $R(\rho)$ *has order* 5 *if and only if* ρ *satisfies Equation* (V) *or* (VI)*.*
(7) $R(\rho)$ *has order* 10 *if and only if* ρ *satisfies Equation* (VII) *or* (VIII)*.*

Proof. If $R(\rho)$ has order 1, then its rotation angle ϕ satisfies $\cos(\phi) = 1$. Thus Equation (2.25) yields $\rho = (\varepsilon, 0, 0, 0)$ where ε is a unit of $\mathbb{Z}[\tau]$, because ρ is $\mathbb{Z}[\tau]$-primitive. Conversely, one verifies $R\left((\varepsilon, 0, 0, 0)\right) = E_3$ by (2.11) for every unit ε of $\mathbb{Z}[\tau]$. Statements (2) to (5) follow as in Lemma 2.38. To show (6), assume that $R(\rho)$ has order 5. Simple calculations for its rotation angle ϕ show that $\phi \in \{2\pi/5, 4\pi/5, 6\pi/5, 8\pi/5\}$. Using (2.25), one obtains for $\phi = 2\pi/5$ and $\phi = 8\pi/5$ that

$$\frac{\tau - 1}{2} = \cos(\phi) = \frac{\kappa^2 - \lambda^2 - \mu^2 - \nu^2}{\kappa^2 + \lambda^2 + \mu^2 + \nu^2}.$$

This is equivalent to Equation (V). For $\phi = 4\pi/5$ and $\phi = 6\pi/5$, one has

$$-\frac{\tau}{2} = \cos(\phi) = \frac{\kappa^2 - \lambda^2 - \mu^2 - \nu^2}{\kappa^2 + \lambda^2 + \mu^2 + \nu^2},$$

which is equivalent to Equation (VI). If on the other hand ρ satisfies Equation (V) and $\psi \in [0, 2\pi)$ denotes the rotation angle of $R(\rho)$, then $(\tau - 1)/2 = \cos(\psi)$ by (2.25). Hence $\psi = 2\pi/5$ or $\psi = 8\pi/5$. And if ρ fulfils Equation (VI), one similarly finds $-\tau/2 = \cos(\psi)$ and thus $\psi = 4\pi/5$ or $\psi = 6\pi/5$. This shows that $R(\rho)$ has order 5.

Finally, let the rotation $R(\rho)$ have order 10. Then its rotation angle ϕ is an element of $\{\pi/5, 3\pi/5, 7\pi/5, 9\pi/5\}$. Thus either $\cos(\phi) = \tau/2$ or $\cos(\phi) = (1 - \tau)/2$. Now Equation (2.25) implies Equation (VII) in the first, and Equation (VIII) in the second case. If we conversely denote by ψ the rotation angle of $R(\rho)$ and ρ satisfies Equation (VII) or (VIII), one obtains $\cos(\psi) = \tau/2$ or $\cos(\psi) = (1 - \tau)/2$ by (2.25). This yields $\psi \in \{\pi/5, 9\pi/5\}$ or $\psi \in \{3\pi/5, 7\pi/5\}$ and hence $R(\rho)$ has order 10. □

Theorem 2.48. *The nontrivial finite generalised dihedral subgroups of* $\mathrm{SO}(3, \mathbb{Q}(\tau))$ *are precisely*

the cyclic group C_k with $k \in \{2, 3, 4, 5, 6, 10\}$,
the dihedral group D_ℓ of order 2ℓ with $\ell \in \{2, 3, 4, 5, 6, 10\}$
and the symmetric group S_4.

Proof. By Theorem 2.42, the symmetric group S_4 is a generalised dihedral group of $\mathrm{SO}(3, \mathbb{Q}) \subset \mathrm{SO}(3, \mathbb{Q}(\tau))$ as are the cyclic groups C_2, C_3, C_4, C_6 and the dihedral groups D_2, D_3, D_4 and D_6. As a consequence of Theorem 2.23 and Lemma 2.45, it remains to show that C_5, C_{10}, D_5 and D_{10} are generalised dihedral subgroups of $\mathrm{SO}(3, \mathbb{Q}(\tau))$.

The $\mathbb{Z}[\tau]$-primitive quaternion $\rho_1 = (\tau + 1, \tau, 0, 1)$ satisfies Equation (V), which implies that $R(\rho_1) \in \mathrm{SO}(3, \mathbb{Q}(\tau))$ has order 5 by Lemma 2.47(6). Hence, $R(\rho_1)$ generates C_5 and one can view this as a generalised dihedral group by adding to $R(\rho_1)$ a second redundant generator given by the rotation by 2π about an orthogonal axis. Similarly, one verifies that $\rho_2 = (3\tau + 1, \tau, 0, 1)$ is a solution of Equation (VII). Therefore, $R(\rho_2)$ has order 10 due to Lemma 2.47(7) and C_{10} is a generalised dihedral subgroup of $\mathrm{SO}(3, \mathbb{Q}(\tau))$.

For $\ell \in \{5, 10\}$, one has $G_{1/4}(\ell, 2) \simeq D_\ell$ (cf. p. 34). Consider $\rho' = (0, 0, 1, 0)$. By Lemma 2.47(2), $R(\rho') \in \mathrm{SO}(3, \mathbb{Q}(\tau))$ has order 2. Since the rotation axis of $R(\rho')$ is given by $(0,1,0)^t$, one easily observes that it is orthogonal to the rotation axis of $R(\rho_1)$ and $R(\rho_2)$, respectively, which is given by $(\tau,0,1)^t$. Thus, both D_5 generated by $R(\rho_1)$ and $R(\rho')$, as well as D_{10} generated by $R(\rho_2)$ and $R(\rho')$ are generalised dihedral subgroups of $\mathrm{SO}(3,\mathbb{Q}(\tau))$. $\square$

REMARK 2.49. The above list of generalised dihedral subgroups of $\mathrm{SO}(3,\mathbb{Q}(\tau))$ is not exhausting. In contrast to the rational case, $\mathrm{SO}(3,\mathbb{Q}(\tau))$ contains infinite generalised dihedral groups. For example, $G_{1/4}(6,4) \simeq D_6 \star_{D_2} D_4$ is infinite by Lemma 2.17 and it is generated by $R(\rho)$ and $R(\rho')$, where $\rho = (2\tau,-1,\tau,\tau+1)$ and $\rho' = (3\tau,\tau+1,0,1)$. Namely, from $\rho,\rho' \in \mathbb{H}(\mathbb{Q}(\tau))$ follows $R(\rho), R(\rho') \in \mathrm{SO}(3,\mathbb{Q}(\tau))$ and since ρ is a solution of Equation (I), Lemma 2.47(3) implies that $R(\rho)$ has order 4. Similarly, $R(\rho')$ has order 6 by Lemma 2.47(4). Finally, one easily observes that $R(\rho)$ and $R(\rho')$ have orthogonal rotation axes, which completes the argument. Note that $G_{1/4}(6,4)$ is the group of orientations of a quaquaversal tiling, cf. Remark 2.32.

Using Lemma 2.47 again, one verifies that other examples are $C_4 \star C_3 \simeq G_{1/12}(4,3)$ generated by $R(\rho_1)$ and $R(\rho_1')$, where $\rho_1 = (2\tau,1,\tau,\tau+1)$ and $\rho_1' = (\tau,1,0,\tau+1)$, and $D_5 \star_{C_2} C_4 \simeq G_{2/5}(4,4)$ generated by $R(\rho_2)$ and $R(\rho_1)$, where $\rho_2 = (1,1,0,0)$, to mention only a few. However, the complete classification of all generalised dihedral subgroups of $\mathrm{SO}(3,\mathbb{Q}(\tau))$ remains an open problem so far.

We considered $\mathrm{SO}(3,\mathbb{Q}(\tau))$ motivated by the connection with the standard icosahedral modules. When studying 3-dimensional quasicrystals with 8- or 12-fold rotational symmetry, the fields $\mathbb{Q}(\sqrt{2})$ and $\mathbb{Q}(\sqrt{3})$ instead of $\mathbb{Q}(\tau)$ may occur. They are the maximal real subfields of the 8th and 12th cyclotomic field, respectively. Both $\mathbb{Q}(\sqrt{2})$ and $\mathbb{Q}(\sqrt{3})$ are quadratic fields whose rings of integers $\mathcal{O}_{\mathbb{Q}(\sqrt{2})} = \mathbb{Z}[\sqrt{2}]$ and $\mathcal{O}_{\mathbb{Q}(\sqrt{3})} = \mathbb{Z}[\sqrt{3}]$ are principal ideal domains. Thus they both have class number one; cf. **[42, 28]** for details. By Theorem 2.35, $\mathrm{SO}(3,\mathbb{Q}(\sqrt{2}))$ is parametrised by the set of $\mathbb{Z}[\sqrt{2}]$-primitive quaternions whereas $\mathrm{SO}(3,\mathbb{Q}(\sqrt{3}))$ is parametrised by the set of $\mathbb{Z}[\sqrt{3}]$-primitive quaternions. Analogously to the line of reasoning for the case $\mathbb{Q}(\tau)$, one can show the following.

THEOREM 2.50. *The nontrivial finite generalised dihedral subgroups of* $\mathrm{SO}(3,\mathbb{Q}(\sqrt{2}))$ *are precisely*

the cyclic group C_k with $k \in \{2,3,4,6,8\}$,
the dihedral group D_ℓ of order 2ℓ with $\ell \in \{2,3,4,6,8\}$
and the symmetric group S_4.

Moreover, the nontrivial finite generalised dihedral subgroups of $\mathrm{SO}(3,\mathbb{Q}(\sqrt{3}))$ *are exactly comprised of*

the cyclic group C_k with $k \in \{2,3,4,6,12\}$,
the dihedral group D_ℓ of order 2ℓ with $\ell \in \{2,3,4,6,12\}$
and the symmetric group S_4.

$\square$

As expected, the groups C_8 and D_8 show up in $\mathrm{SO}(3,\mathbb{Q}(\sqrt{2}))$, while C_{12} and D_{12} do in $\mathrm{SO}(3,\mathbb{Q}(\sqrt{3}))$, in line with the above mentioned motivation.

Outlook

As pointed out in the first chapter, the situation regarding coincidence rotations in 3-dimensions is considerably more complicated than in the 2-dimensional case, because $\mathrm{SO}(3,\mathbb{R})$ is not Abelian in contrast to $\mathrm{SO}(2,\mathbb{R})$. It is generally desirable to gain further knowledge on the group structure of coincidence rotations of modules in dimensions $d \geqslant 3$.

Theorem 1.23 determines the structure of the factor group of similarity modulo coincidence isometries for certain modules as the direct sum of cyclic groups of prime power orders. Hereby, the theoretical framework was set for the structure of the factor group of an important class of modules (containing relevant examples in quasicrystallography). In two dimensions, there are examples where the factor group is the direct sum of countably infinitely many cyclic groups (cf. Section 1.2). Recent results [**44**] however suggest that the situation might be simpler for odd dimensions. The question that arises is which groups actually do appear as factor groups in practice.

The proof of our main result of the second chapter (Theorem 2.42) heavily relies on the findings of [**37**], where the approach is as follows. Find presentations of the groups $G_{n/m}(p,q)$ by showing that the simple relations induced by (2.6) to (2.8) are the only ones in that group. Interestingly, these relations all derive from the relations of the rotation symmetry group of the cube. However, the rotation symmetry group of some Platonic solids do not appear among the generalised dihedral groups, namely the tetrahedral and the icosahedral group. They each can be generated by a pair of finite order rotations, but only about axes separated by an irrational angle (cf. Lemma 2.26). Hence rational angles are not sufficient to reach the icosahedral group, which is of interest regarding icosahedral quasicrystals and particularly due to the fact that it is the group of orientation-preserving coincidence isometries of the standard icosahedral modules. So a natural next step is this: To what extent do we have to generalise our setting in order to accommodate the icosahedral group, i.e. what type of angles α, as a generalisation of rational multiples of π, suffice to express the icosahedral group as $G_\alpha(p,q)$? In many cases, the angles between rotation axes of two generators of the icosahedral group are irrational angles α for which $e^{2i\alpha}$ is a quadratic irrational. Such angles are called geodetic in [**16**]. The icosahedral group can for example be generated by two rotations of order 5 such that the angle between the two corresponding rotation axes is geodetic (cf. Example 2.27).

Classifying such groups with the same methods as before involves determining exactly when there are relations between two rotations of finite order about axes that enclose a geodetic angle. This was the task of [**16**]. In fact, there is only a finite set of geodetic angles that can support nontrivial relations between two finite order rotations, cf. [**16**, Thm. 1]. The necessary conditions are known, but, except for the case where the two rotations both have prime order, it is not known which of the geodetic angles actually do support nontrivial relations. According to [**16**], it is a "daunting task" to actually determine the relations, if they exist. In light of these explanations, a further generalisation probably needs new methods. Here, the challenging questions can be summarised as follows.

- What are the relations induced by geodetic angles?
- What is the structure of the subgroups $G_\alpha(p, q)$ of $\mathrm{SO}(3, \mathbb{R})$ that are generated by two rotations of finite order about axes separated by a geodetic angle α?

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